Linear Programming

Graphical and Computer Methods

Linear Programming (LP)

- **Managers continually plan and make decisions related to resource allocation**
  - **Resources typically include labor, raw materials, money, time, warehouse space, machinery, etc.**
  - **Management's goal: use available resources (all of which cost money) as efficiently as possible**
- **Linear programming is a widely used <u>mathematical technique</u> designed to help management with resource allocation decisions**
  - **Microsoft Excel Solver**

Linear Programming

- **A mathematical method for determining the best way to accomplish an objective**
  - **Minimize cost, maximize profit, etc.**
    - » **Maximum profit production plans**
    - » **Minimum cost blends of chemicals/petroleum/foods**
    - » **Minimum cost transportation of goods/materials**
    - » **Minimum cost personnel scheduling**
    - » **Maximum profit cash flow allocation**
- **All LP problems have <u>four common properties,</u> <u>three major components</u>, and <u>five basic assumptions</u>**

Four Common Properties

- **All problems seek to maximize or minimize some quantity**
  - ***Objective function*** **usually maximizes profit or minimizes cost**
- **Restrictions or constraints are present which limit the degree to which an objective can be achieved**
- **Alternative courses of action are available to choose from**
- **Objectives and constraints are expressed in terms of linear equations or inequalities**

Three Major Components

- **Decision Variables**
- **Objective Function**
- **Resource Constraints**

Decision Variables

What Can I Adjust?

- **Mathematical symbols that represent levels of activity by a firm**
  - **How much of a product should be produced, how much money should be invested, etc.**

- **“Adjustable Cells”**

Objective Function

What Do I Mean by Best?

- **A <u>linear</u> mathematical relationship that describes the relationship in terms of the decision variables**
  - **Maximize profit, kills/sortie**
  - **Minimize cost, attrition**

$$Z = aX_1 + bX_2 \qquad\qquad Z = aX_1^2 + b$$

Resource Constraints

What Constraints Must I Obey?

- Linear relationships of the decision variables that represent limits placed on a firm
  - Equalities or inequalities
- Consist of decision variables and parameters
  - Numerical values in the objective function and constraints

$3X_1 + 2X_2 \leq 75$ $2X_1 + 5X_2 \geq 110$

LP Assumptions

- Parameter Availability
- Proportionality
- Additivity
- Divisibility
- Nonnegativity

Parameter Availability

- **Model parameters must be known or reliably estimated**

- **Model parameters do not change during the period being studied**
  - **Objective function coefficients**
  - **Limits on resource availability**
  - **Constraint coefficients**

Proportionality

- **Contributions to value of objective function and the amount of resources used are proportional to the value of each decision variable**
  - **Doubling the amount of labor used doubles labors contribution to profit**
  - **Constant slope**
    - » **no fixed costs, economies of scale, startup costs**

“If producing one unit of a product requires 3 hours of labor, producing ten units requires 30 hours of labor”

Additivity

- **Total value of the objective function = sum of contributions of each decision variable**

- **Total resources used = sum of resources used for each activity**

"The sum of all activities equals the sum of the individual activities"

Divisibility

- **Solutions are not restricted to whole numbers**
  - **Continuous and may take any fractional value**

- **If a fractional result can not be produced, an *integer programming* problem exists**

Nonnegativity

- **Assume that all answers and variables are nonnegative**
  - **Negative values of physical quantities are impossible**

Model Formulation:
A Product Mix Problem

<u>The Flair Furniture Company</u>

- **Produces inexpensive tables and chairs**
  - **Two primary resources required during production:**
    - » **labor hours for carpentry**
    - » **labor hours for painting and varnishing**

- **How many tables and chairs should be produced each week in order to maximize profit?**
  - **Resource requirements, limitations, and profit per item produced:**

	Hours Required Per Unit		Weekly Hours
Department	**Tables**	**Chairs**	**Available**
Carpentry	**4**	**3**	**240**
Painting/Varnishing	**2**	**1**	**100**
Profit per Unit	**$7**	**$5**	

Decision Variables

What does management control?

Decision Variables

X_1 = # of tables to produce each week

X_2 = # of chairs to produce each week

Objective Function

What is meant by best?

Objective Function

- **Maximize Profit**
  - **Profit per table= \$7, Profit per chair = \$5**
  - **# of tables to produce = X_1, # of chairs to produce = X_2**

- **Profit Function:**

$$Z = \$7X_1 + \$5X_2$$

- **Objective Function:**

$$\underline{\textit{Maximize}} \quad Z = \$7X_1 + \$5X_2$$

Resource Constraints

What constraints must be obeyed?

What resources are limited?

Resource Constraints

Labor for Carpentry

- Available: 240 hours per week

- Required: 4 hours per table produced, 3 hours per chair produced

- Labor constraint: $4X_1 + 3X_2 \leq 240$

Resource Constraints

<u>Labor for Painting and Varnishing</u>

- **Available: 100 hours per week**

- **Required: 2 hours per table produced, 1 hour per chair produced**

- **Labor constraint: $2X_1 + 1X_2 \leq 100$**

Resource Constraints

Non-negativity

- No negative quantities can be produced

$$X_1, X_2 \geq 0$$

Model Summary *

X_1 = # of tables to produce each week

X_2 = # of chairs to produce each week

$$\text{Maximize } Z = \$7X_1 + \$5X_2$$

subject to

$4X_1 + 3X_2 \leq 240$

$2X_1 + 1X_2 \leq 100$

$X_1, X_2 \geq 0$

Graphical Solution

- **Practically limited to problems of two variables**
  - **Provides insight into the solution process for more complex problems**

- **Five-step procedure:**

- **(1) Graph feasible solutions for each constraint**

- **(2) Determine the feasible solution area**

- **(3) Draw an objective function line (*isoprofit line*)**

- **(4) Move parallel isoprofit lines toward better objective function values**

- **(5) Solve for optimum values of X_1 and X_2**

Determining Feasible Solutions

- **Plot each of the problem's constraints on a graph**

$$\boxed{4X_1 + 3X_2 \leq 240}$$

$$2X_1 + 1X_2 \leq 100$$

$$X_1, X_2 \geq 0$$

Determining Feasible Solutions

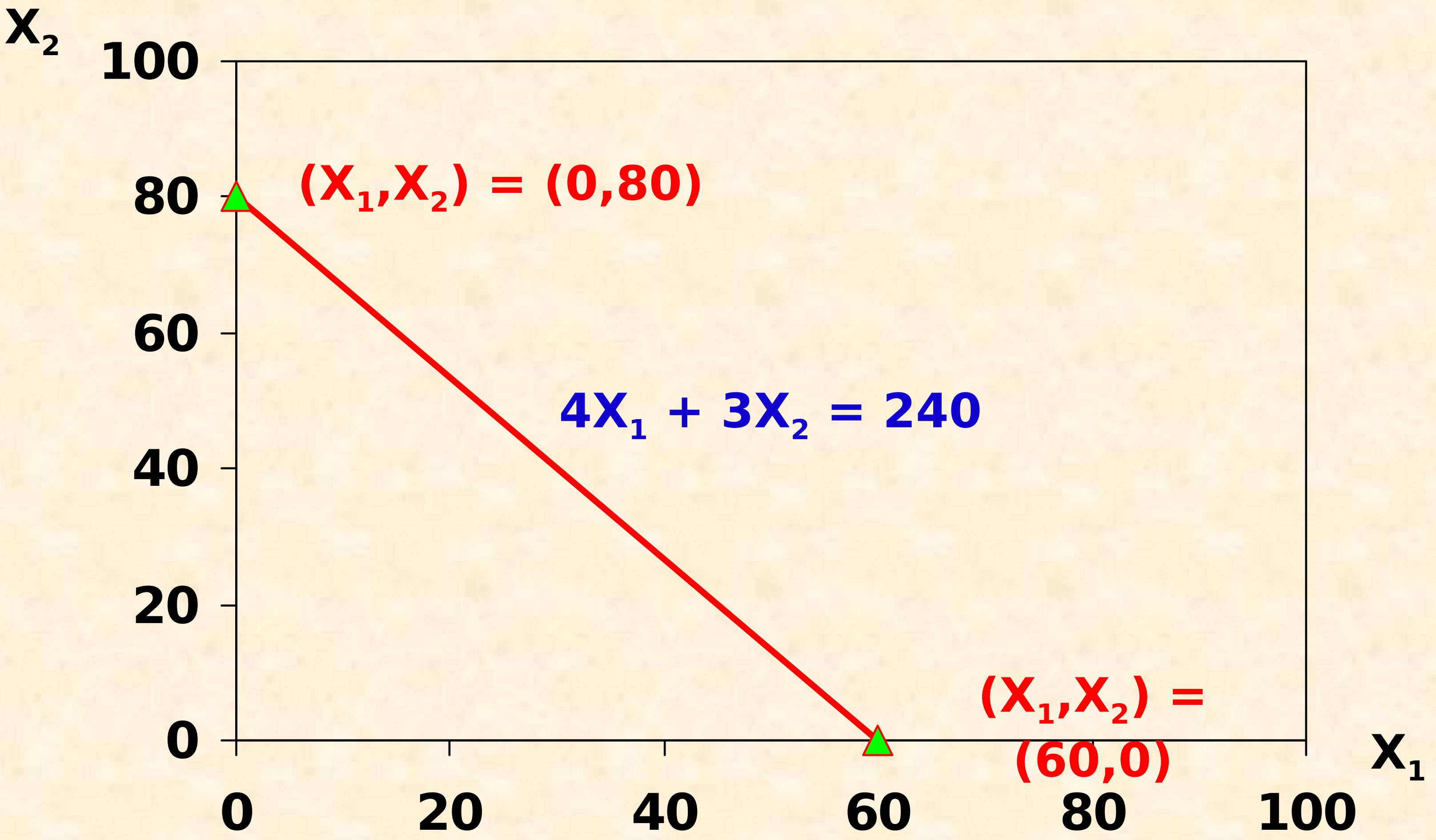
$$4X_1 + 3X_2 \leq 240$$

- To represent graphically, convert to an equality:

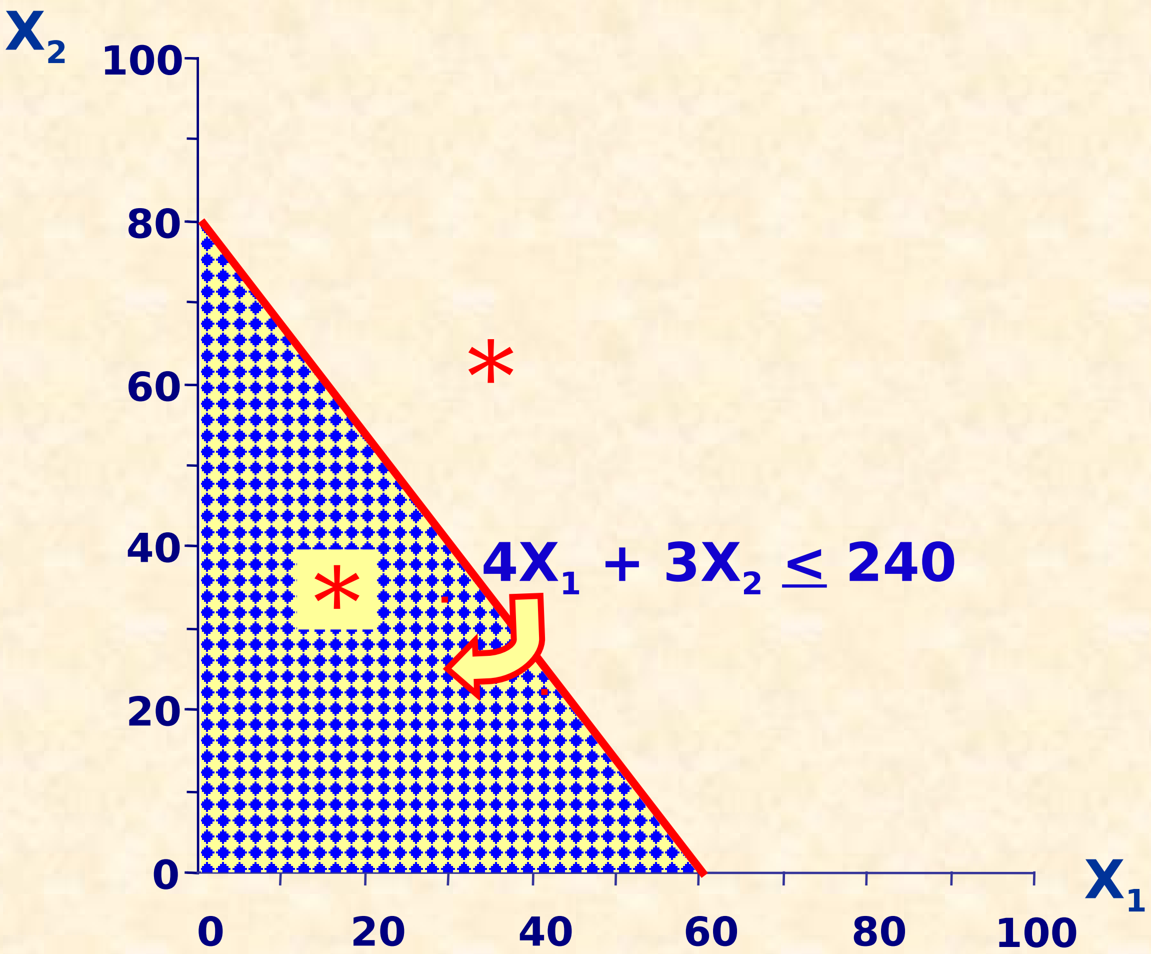
$$4X_1 + 3X_2 = 240$$

- Plot a straight line representing any two points that satisfy the equation
  - Determine one end point of the line by setting one of the two variables equal to zero and solving the resulting equation
    - » set $X_1 = 0$: $4(0) + 3X_2 = 240 \Rightarrow 3X_2 = 240 \Rightarrow X_2 = 80$
  - Repeat for the second variable
    - » set $X_2 = 0$: $4X_1 + 3(0) = 240 \Rightarrow 4X_1 = 240 \Rightarrow X_1 = 60$

Determining Feasible Solutions



Determining Feasible Solutions



Determining Feasible Solutions

Repeat for other constraints

$$4X_1 + 3X_2 \leq 240$$

$$\boxed{2X_1 + 1X_2 \leq 100}$$

$$X_1, X_2 \geq 0$$

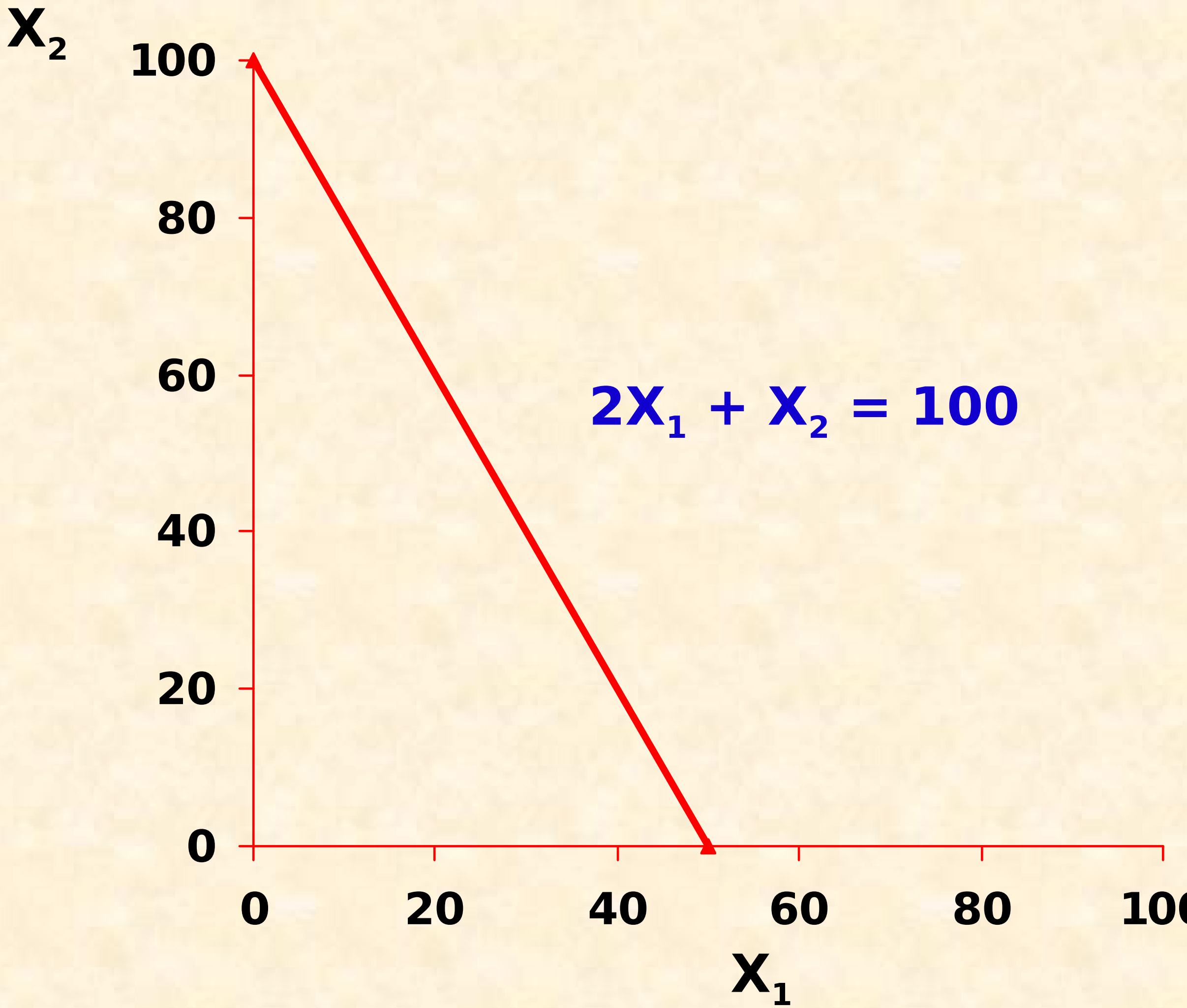
$$2(0) + 1X_2 = 100$$

$$X_2 = 100$$

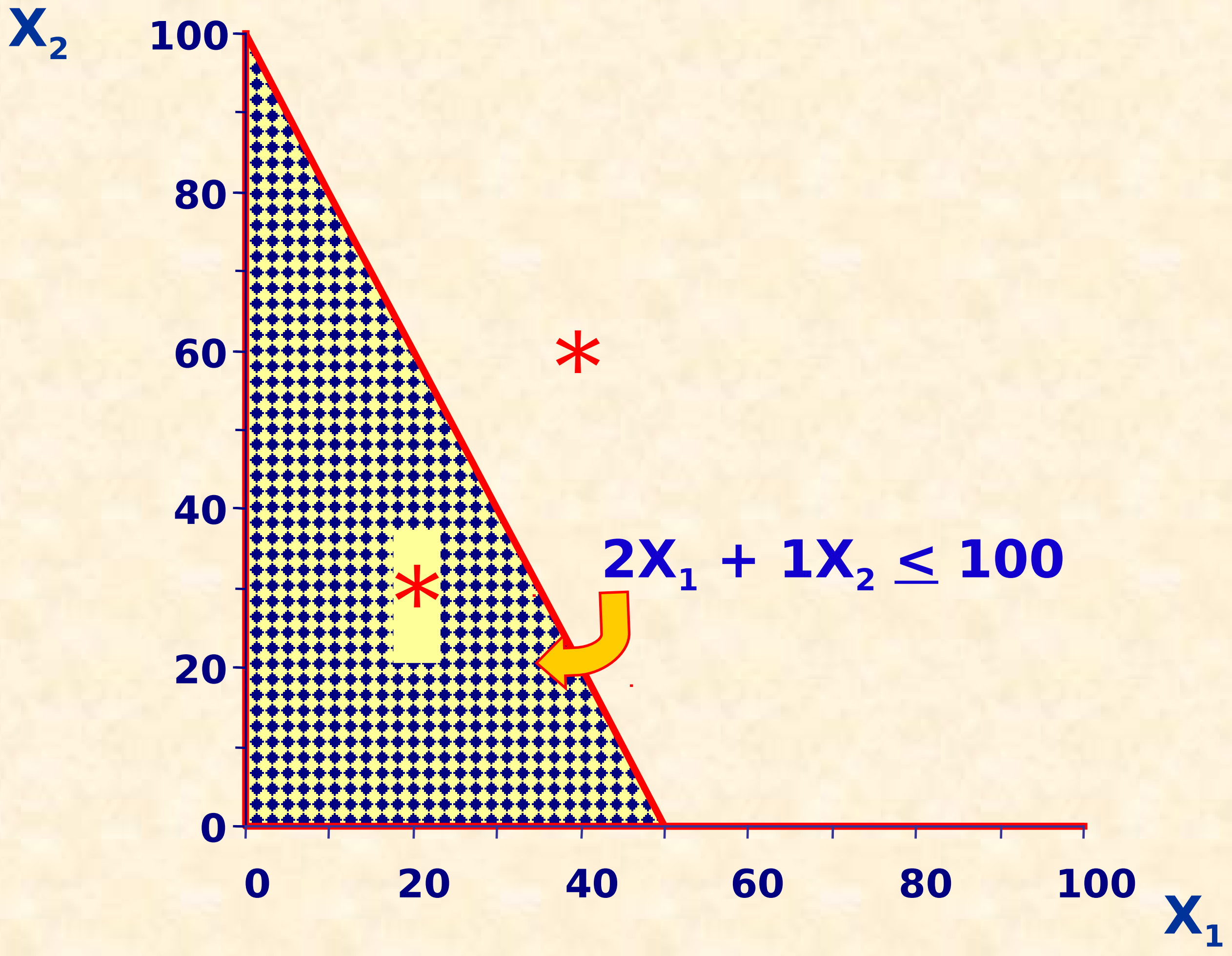
$$2X_1 + 1(0) = 100$$

$$X_1 = 50$$

Determining Feasible Solutions



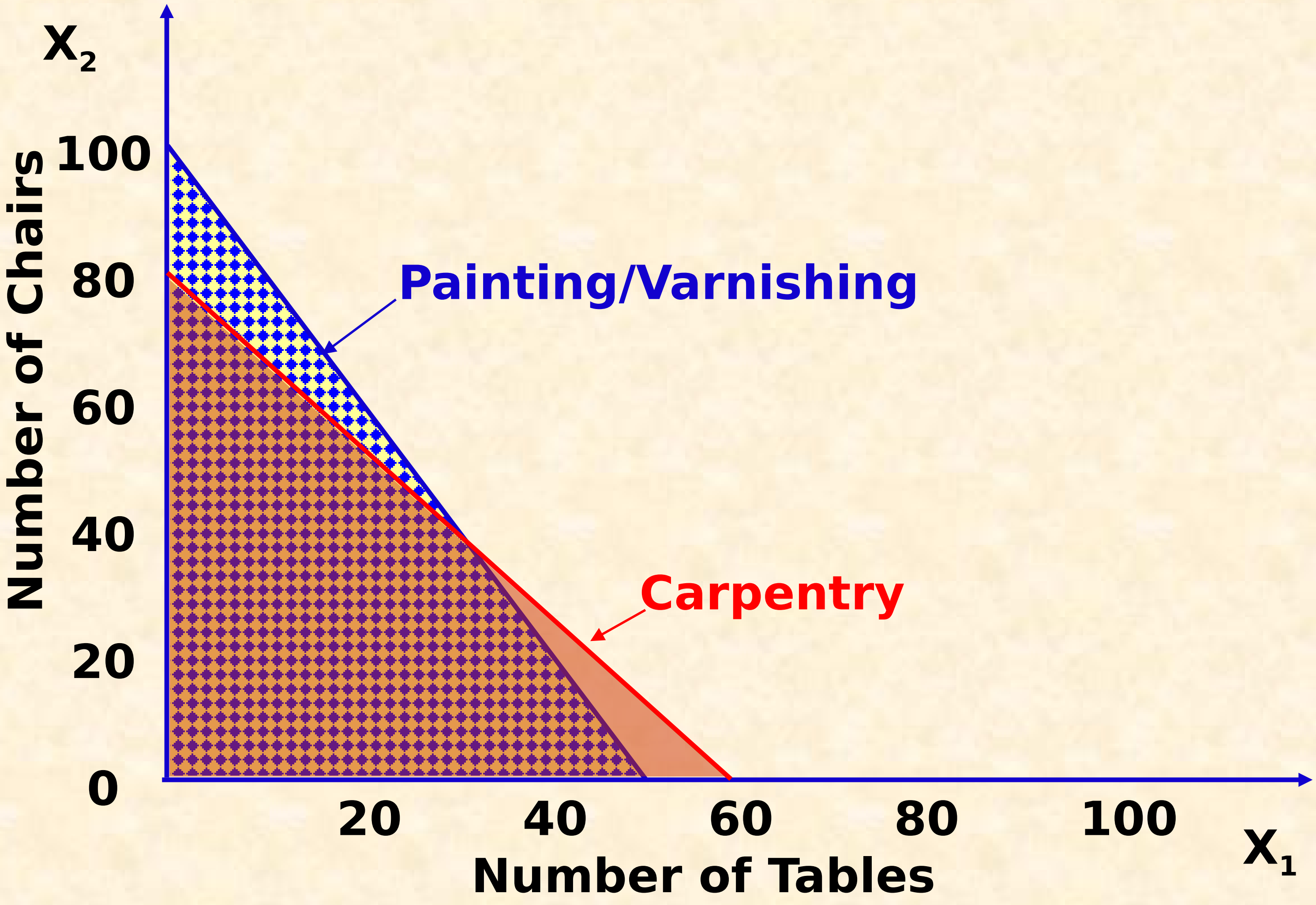
Determining Feasible Solutions



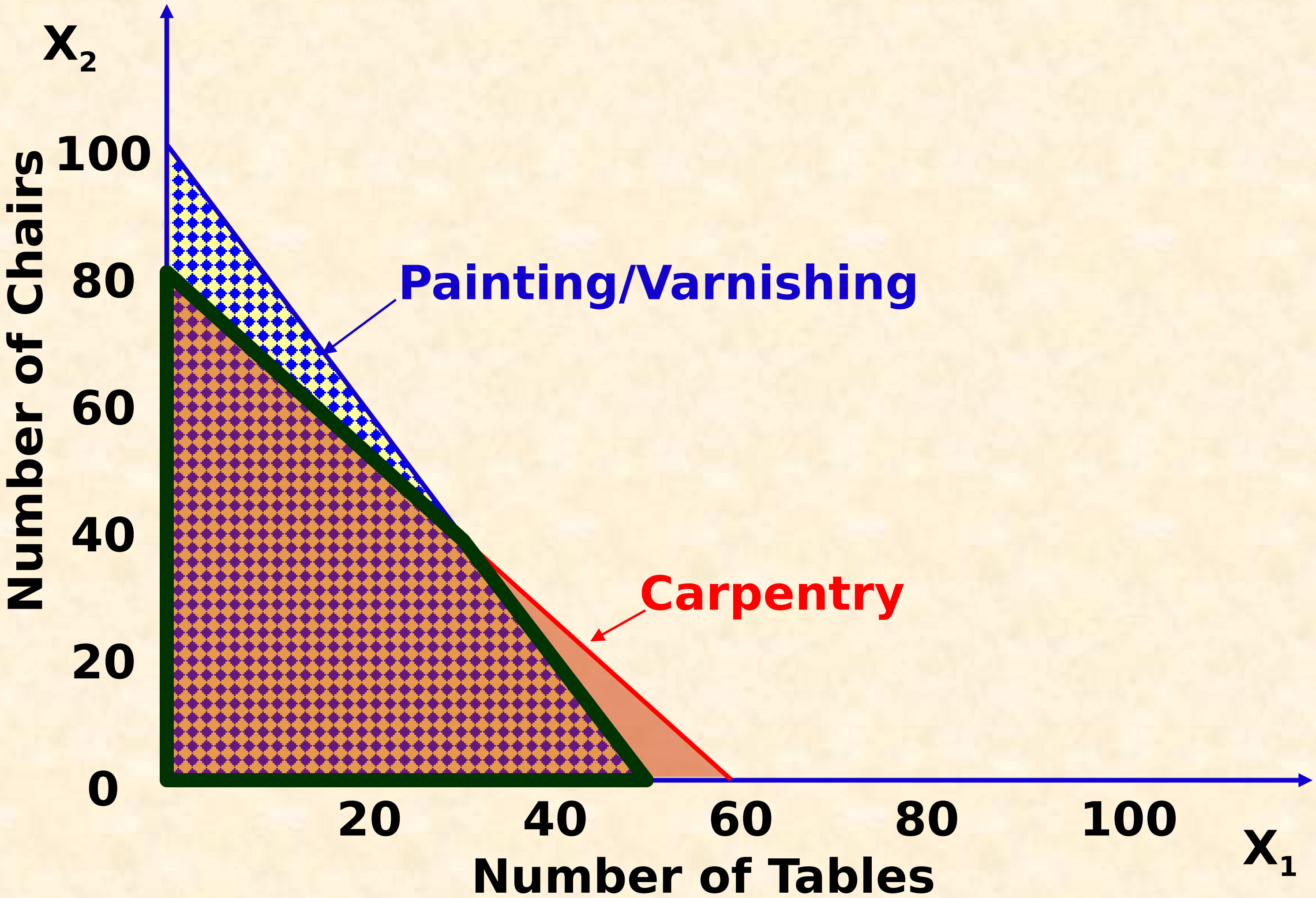
Determine the Feasible Region

- **Find the set of solution points that satisfies all constraints simultaneously**

Determining the Feasible Region



Determining the Feasible Region



Draw an Objective Function Line

The Isoprofit Line Method

- **Optimal solution: the point lying in the feasible region that produces the highest profit**

<u>Finding the optimal solution point</u>

- **Select an arbitrary profit amount**
  - **Profit determined by objective function**
  - **Select a small level of profit (Z) that is a multiple of the product of the objective function coefficients**
    - » **Set $X_1 = 0$, solve for X_2**
    - » **Set $X_2 = 0$, solve for X_1**

end points of an *<u>isoprofit line</u>*

$$Z = 7X_1 + 5X_2$$

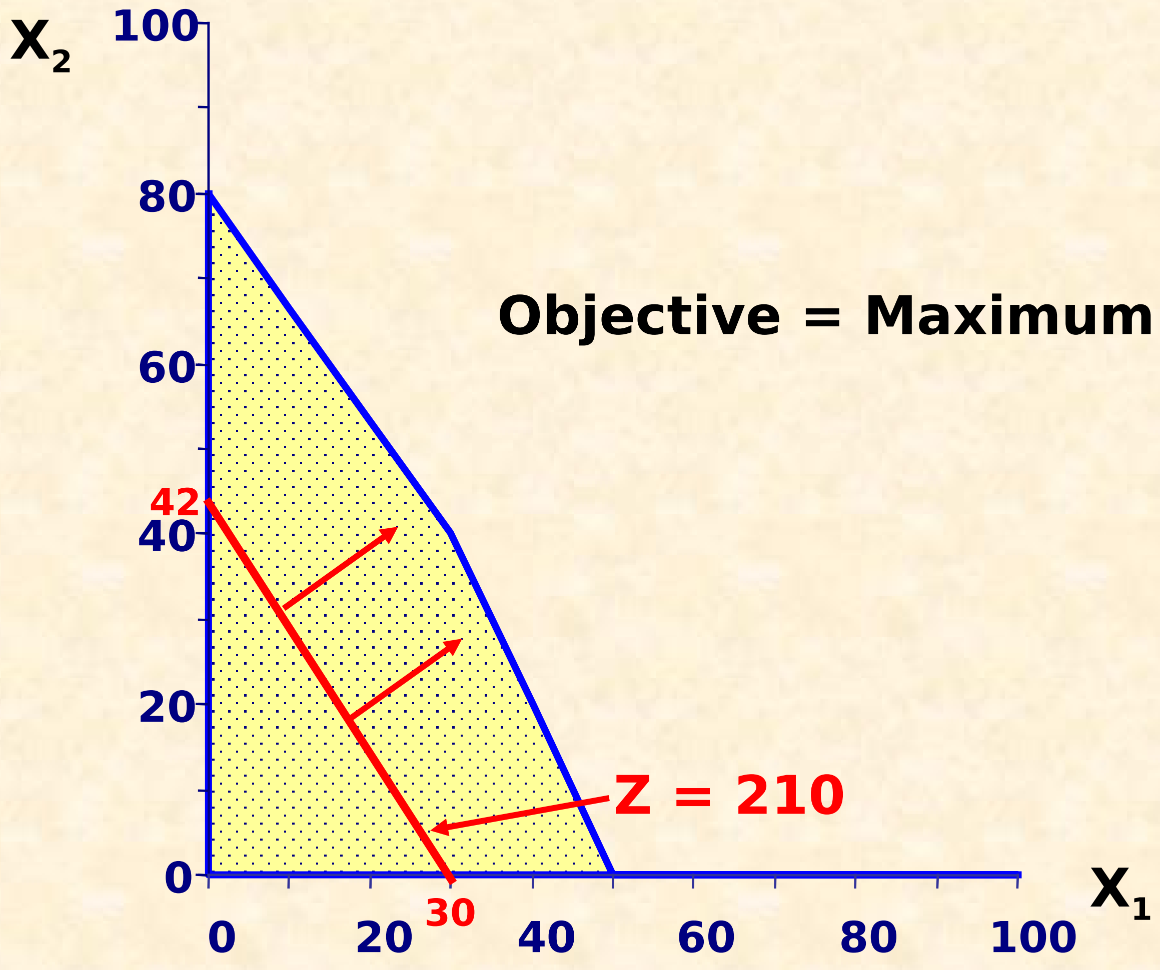
(7)(5) = 35; pick a multiple of 35

Set Z = (6)(35) = 210

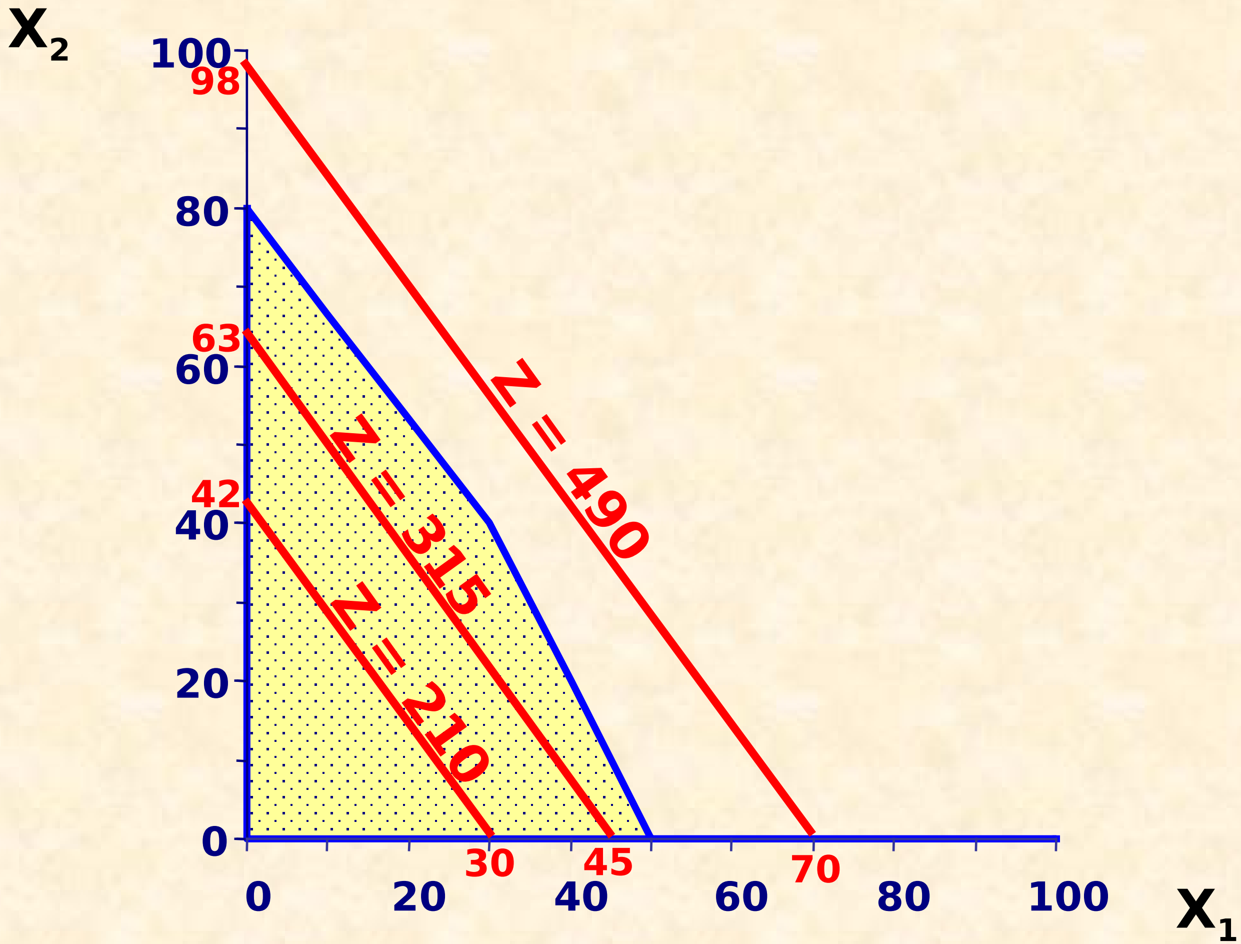
If $X_1 = 0$, $X_2 = 42$

If $X_2 = 0$, $X_1 = 30$

Move Isoprofit Lines Toward **Better** Objective Function Values

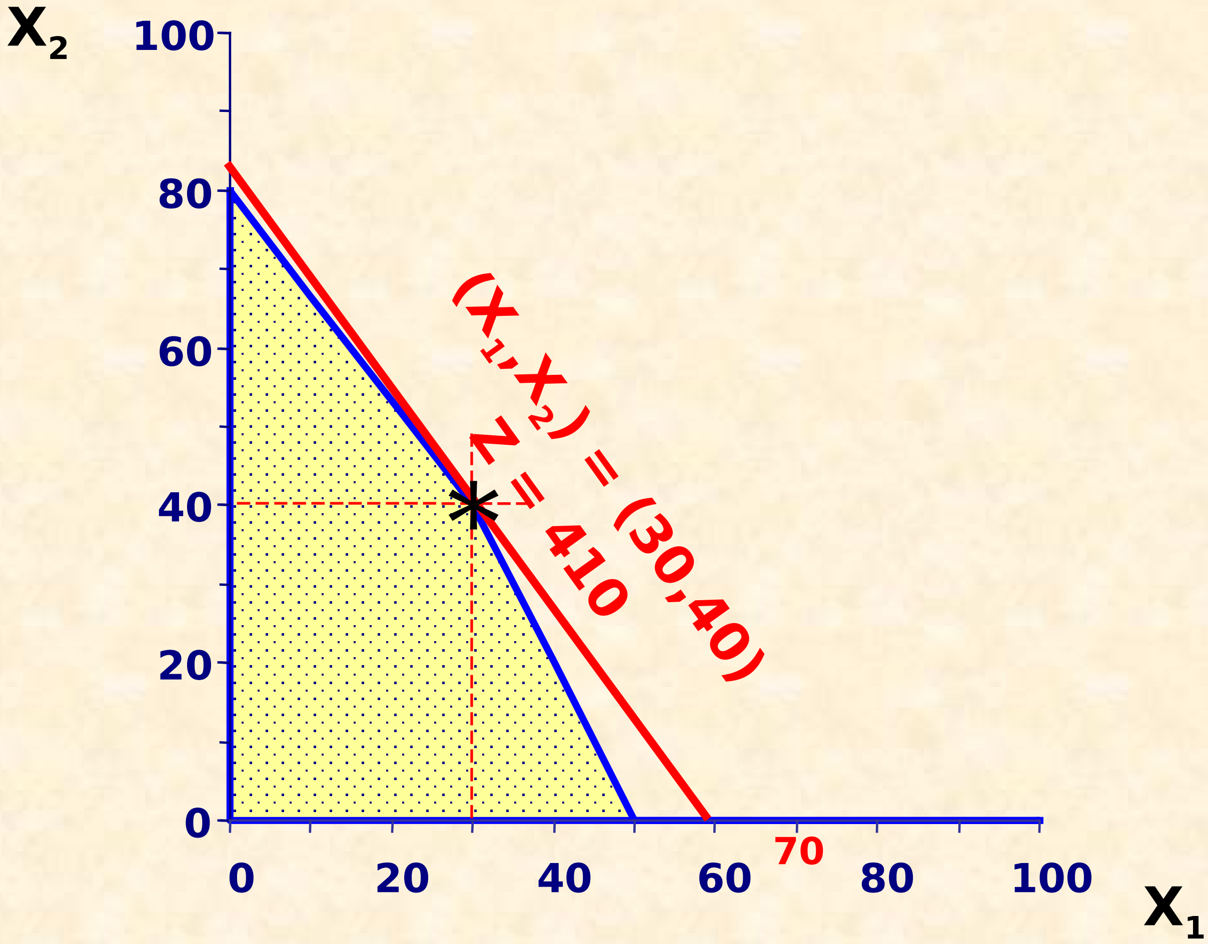


Isoprofit Lines



The highest profit line that still touches a point in the feasible region is the *optimal solution*

Solve for X_1, X_2



The highest profit line touches the tip of the feasible region at $(X_1,X_2) = (30,40)$. Optimal (maximum) profit is \$410.

Finding X_1, X_2 Using the Corner Point Solution Method

- **Based on the mathematical theory that is the foundation of linear programming**
  - **"The optimum solution must lie on the border of the feasible region at one of the *corner points*"**
    - **» involves looking at the profit at every corner point (*extreme* point) of the feasible region**

- **The feasible region of the Flair Furniture Company example is a four-sided polygon**
  - **Four sides = four corner points to be evaluated**

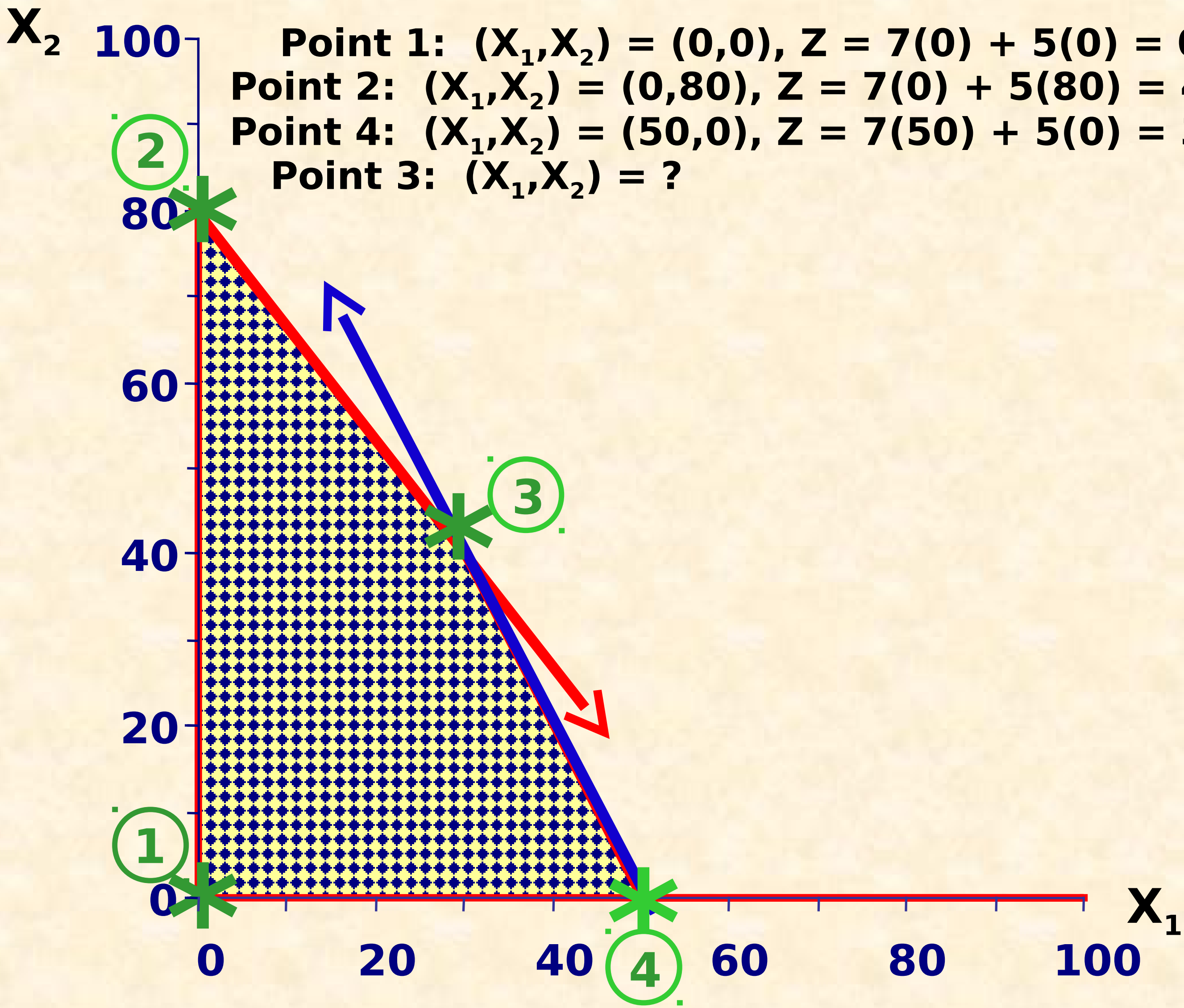
Corner Points

Point 1: $(X_1,X_2) = (0,0)$, $Z = 7(0) + 5(0) = 0$

Point 2: $(X_1,X_2) = (0,80)$, $Z = 7(0) + 5(80) = 400$

Point 4: $(X_1,X_2) = (50,0)$, $Z = 7(50) + 5(0) = 350$

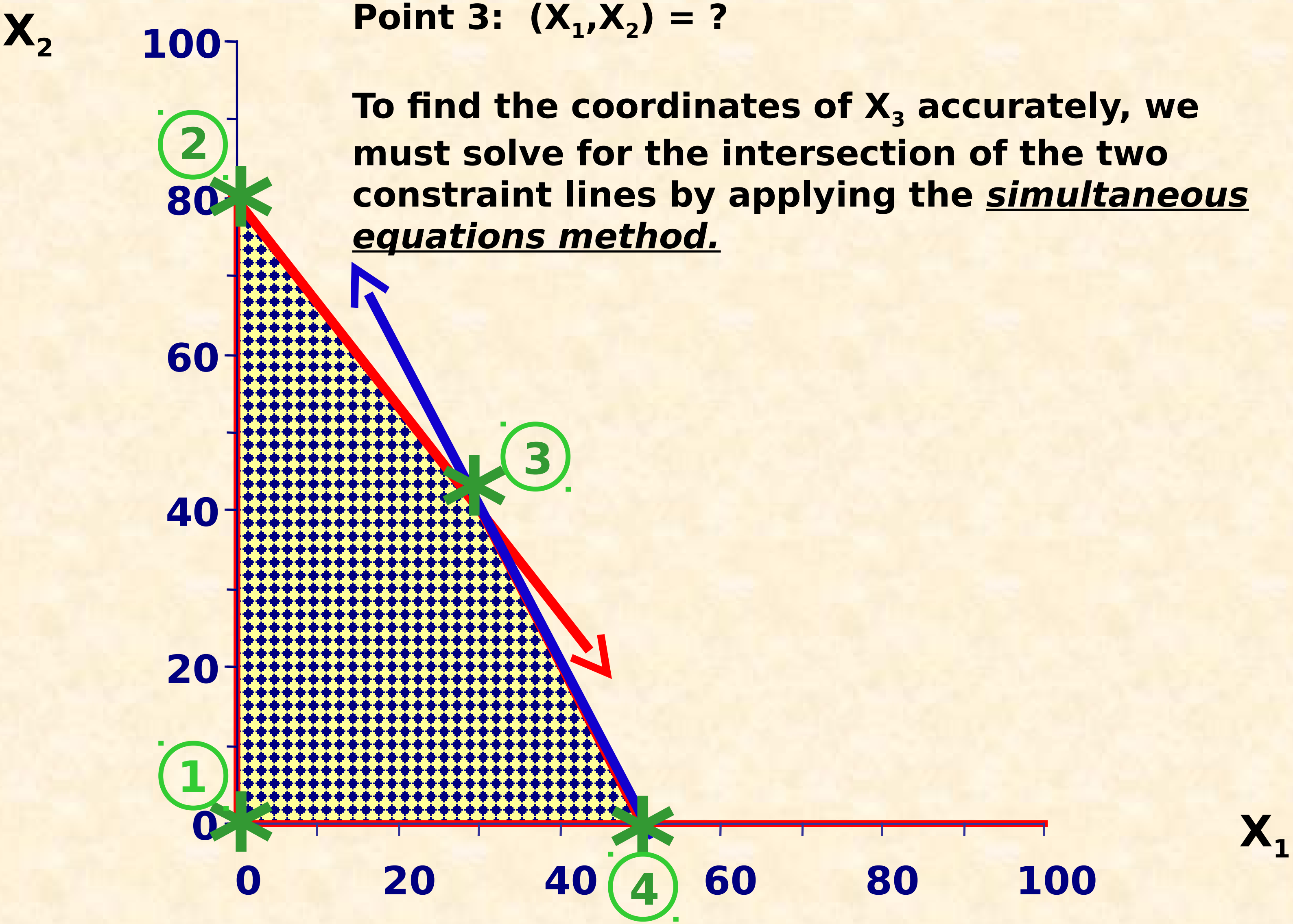
Point 3: $(X_1,X_2) = ?$



Corner Points

Point 3: (X_1,X_2) = ?

To find the coordinates of X_3 accurately, we must solve for the intersection of the two constraint lines by applying the *simultaneous equations method.*



Solving Simultaneous Equations

- First determine which constraint lines intersect at the solution point

$X_1 = 0$ $X_2 = 0$ $4X_1 + 3X_2 = 240$ $2X_1 + 1X_2 = 100$

at the point of intersection, the constraint lines are equal

- Solve both equations for X_1 (or X_2)

$$4X_1 + 3X_2 = 240 \qquad 2X_1 + X_2 = 100$$

$$4X_1 = 240 - 3X_2 \qquad 2X_1 = 100 - X_2$$

$$X_1 = 60 - {}^3/_4 X_2 \qquad X_1 = 50 - {}^1/_2 X_2$$

- Set resulting equations equal and solve for X_2

$$60 - {}^3/_4 X_2 = 50 - {}^1/_2 X_2$$

$$60 - 50 = {}^3/_4 X_2 - {}^1/_2 X_2$$

$$10 = {}^1/_4 X_2$$

$$X_2 = 40$$

Solving Simultaneous Equations

- Substitute optimal value of X_2 into either of the two intersecting resource constraints and solve for X_1

$$4X_1 + 3X_2 = 240$$

$$\text{if } X_2 = 40,$$
$$4X_1 + 3(40) = 240$$
$$4X_1 = 120$$
$$X_1 = 30$$

- Substitute values of X_1, X_2 into objective function to determine optimum (maximum) profit

$$Z = 7X_1 + 5X_2$$
$$= 7(30) + 5(40) = \$410$$

Another Example:

The Beaver Creek Pottery Company

- A Native American small craft operation
  - produce clay bowls and mugs
  - two primary resources: pottery clay and labor
- How many bowls and mugs should be produced each day in order to maximize profit?
  - Resource requirements and profit per item produced:

Product	Labor (hrs/unit)	Clay (lbs/unit)	Profit ($/unit)
Bowl	1	4	40
Mug	2	3	50

- There are 40 hours of labor and 120 pounds of clay available each day.

Decision Variables

What does management control?

Decision Variables

What does management control?

X_1 = # of bowls to produce each day

X_2 = # of mugs to produce each day

Objective Function

What is meant by best?

Objective Function

<u>What is meant by best</u>?

- **Best = Maximum Profit**
  - **Profit per Bowl= \$40, Profit per Mug = \$50**
  - **# of bowls to produce = X_1, # of mugs to produce = X_2**

- **Profit Function:**

$$Z = \$40X_1 + \$50X_2$$

- **Objective Function:**

$$\underline{\textbf{Maximize}} \quad Z = \$40X_1 + \$50X_2$$

Resource Constraints

What constraints must be obeyed?

What resources are limited?

Resource Constraints

Labor

- Available: ?
- Required: ?
- Labor constraint: ?

Resource Constraints

Labor

- Available: 40 hours per day
- Required: 1 hour per bowl produced, 2 hours per mug produced
- Labor constraint: $1X_1 + 2X_2 \leq 40$

Resource Constraints

Pottery Clay

- Available: ?
- Required: ?
- Pottery Clay constraint: ?

Resource Constraints

Pottery Clay

- Available: 120 pounds per day

- Required: 4 pounds per bowl produced, 3 pounds per mug produced

- Pottery Clay constraint: $4X_1 + 3X_2 \leq 120$

Resource Constraints

Non-negativity

- No negative quantities can be produced

- $X_1, X_2 \geq 0$

Model Summary

X_1 = # of bowls to produce each day

X_2 = # of mugs to produce each day

Maximize $Z = \$40X_1 + \$50X_2$

subject to

$1X_1 + 2X_2 \leq 40$

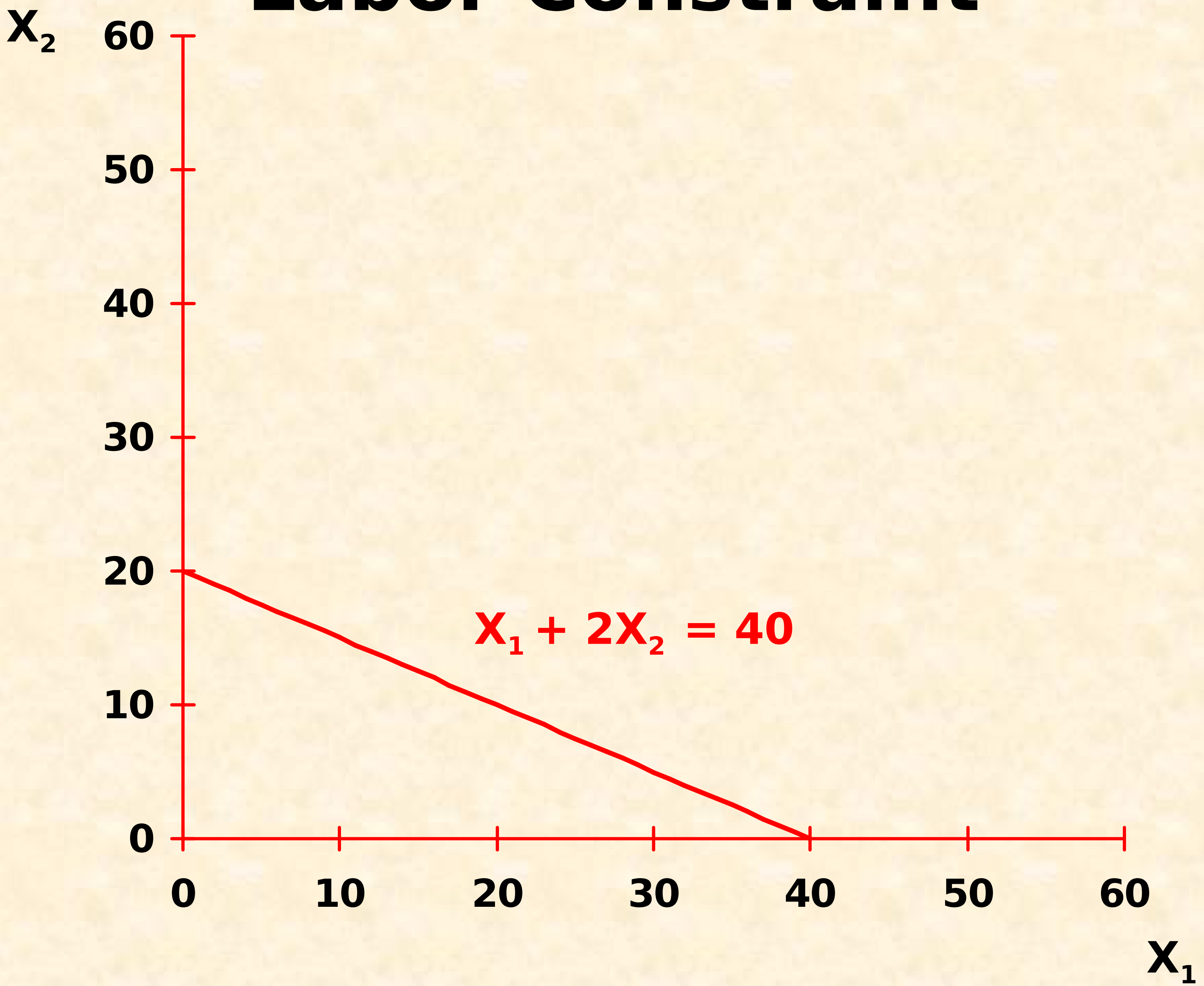
$4X_1 + 3X_2 \leq 120$

$X_1, X_2 \geq 0$

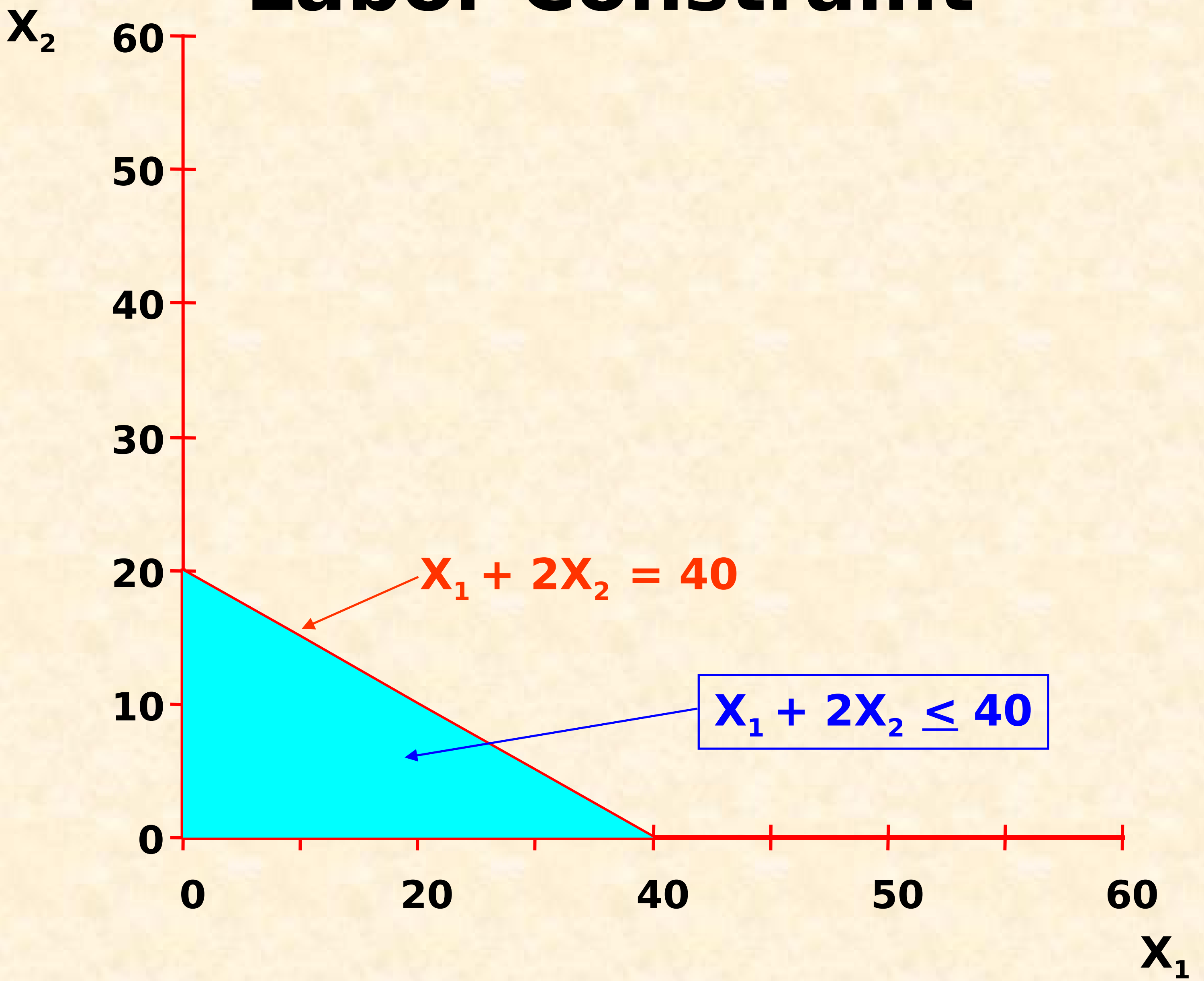
Graph Feasible Solutions

- **Labor Constraint:** $X_1 + 2X_2 \leq 40$
- **Clay Constraint:** $4X_1 + 3X_2 \leq 120$
- **Nonnegativity:** $X_1, X_2 \geq 0$

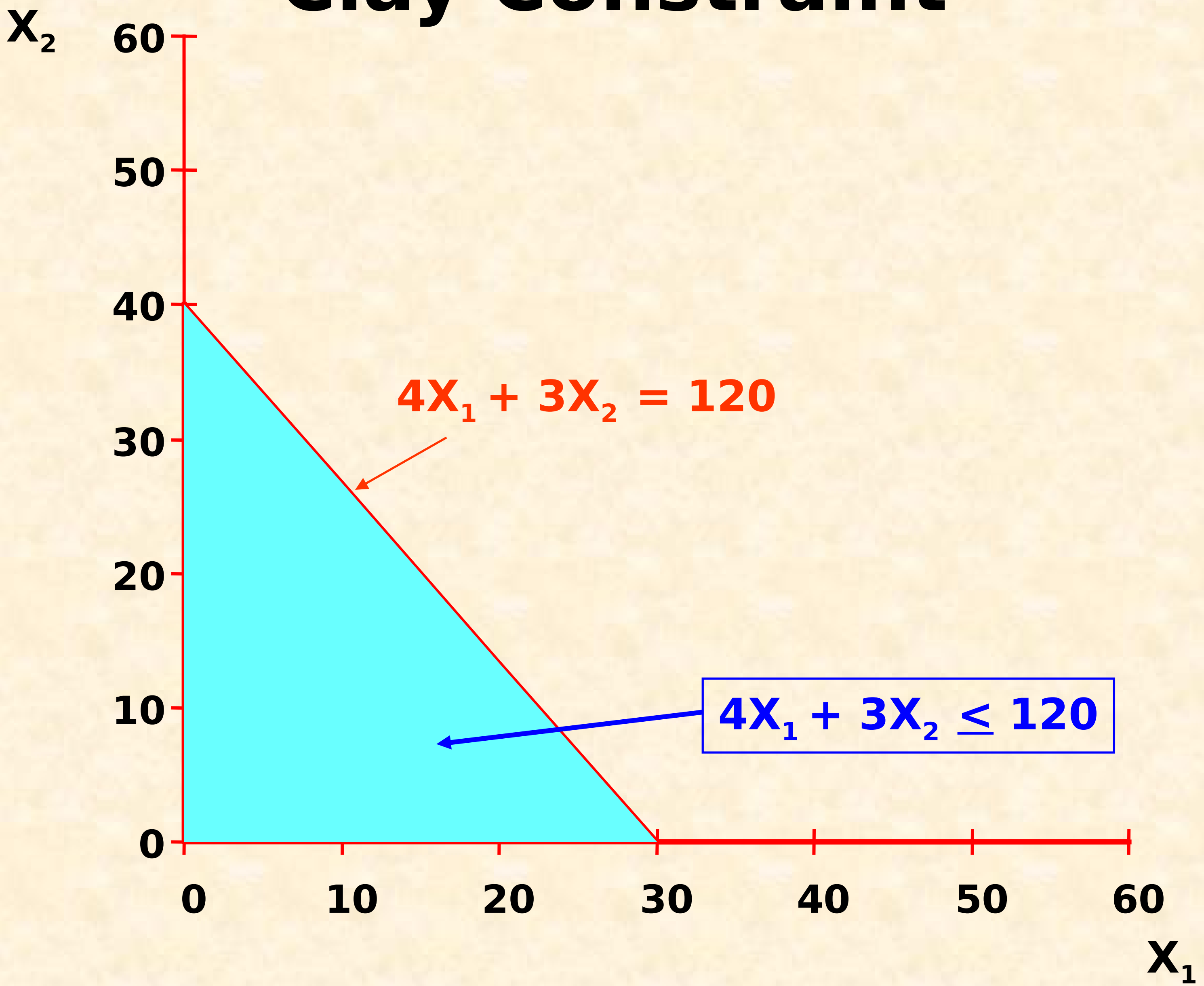
Labor Constraint



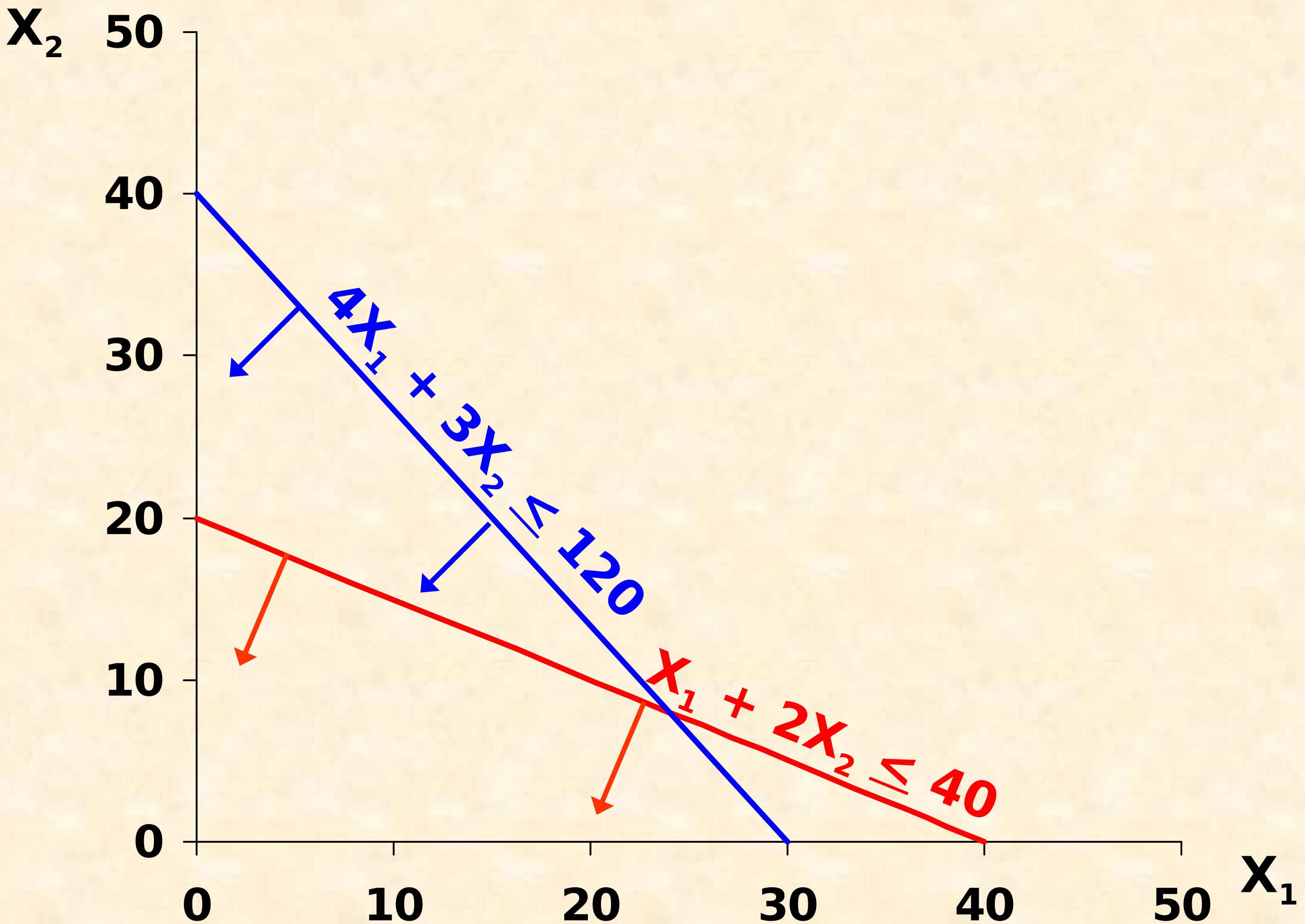
Labor Constraint



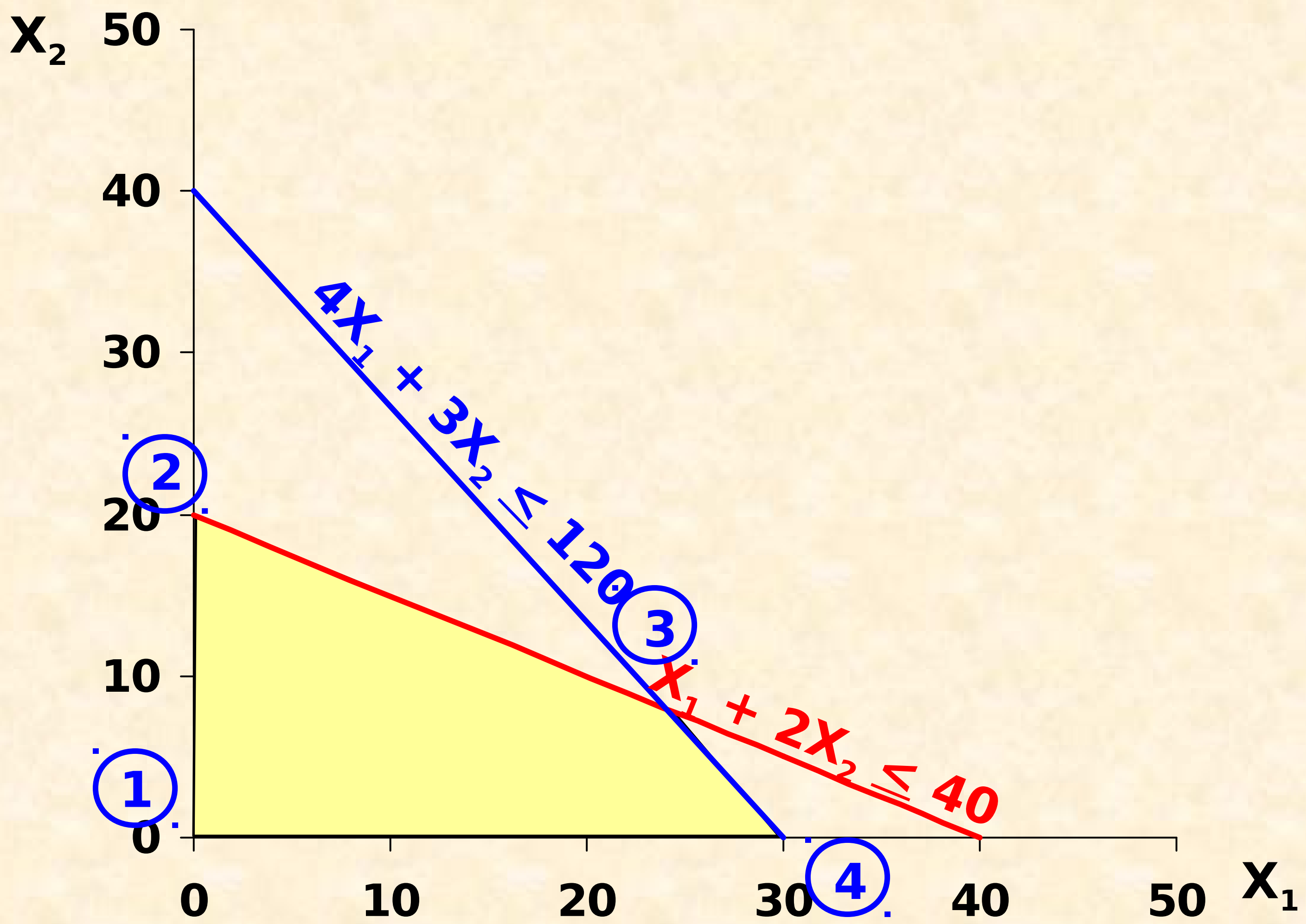
Clay Constraint



Feasible region



Feasible region



Draw Isoprofit Line
Solve for Optimum X_1, X_2

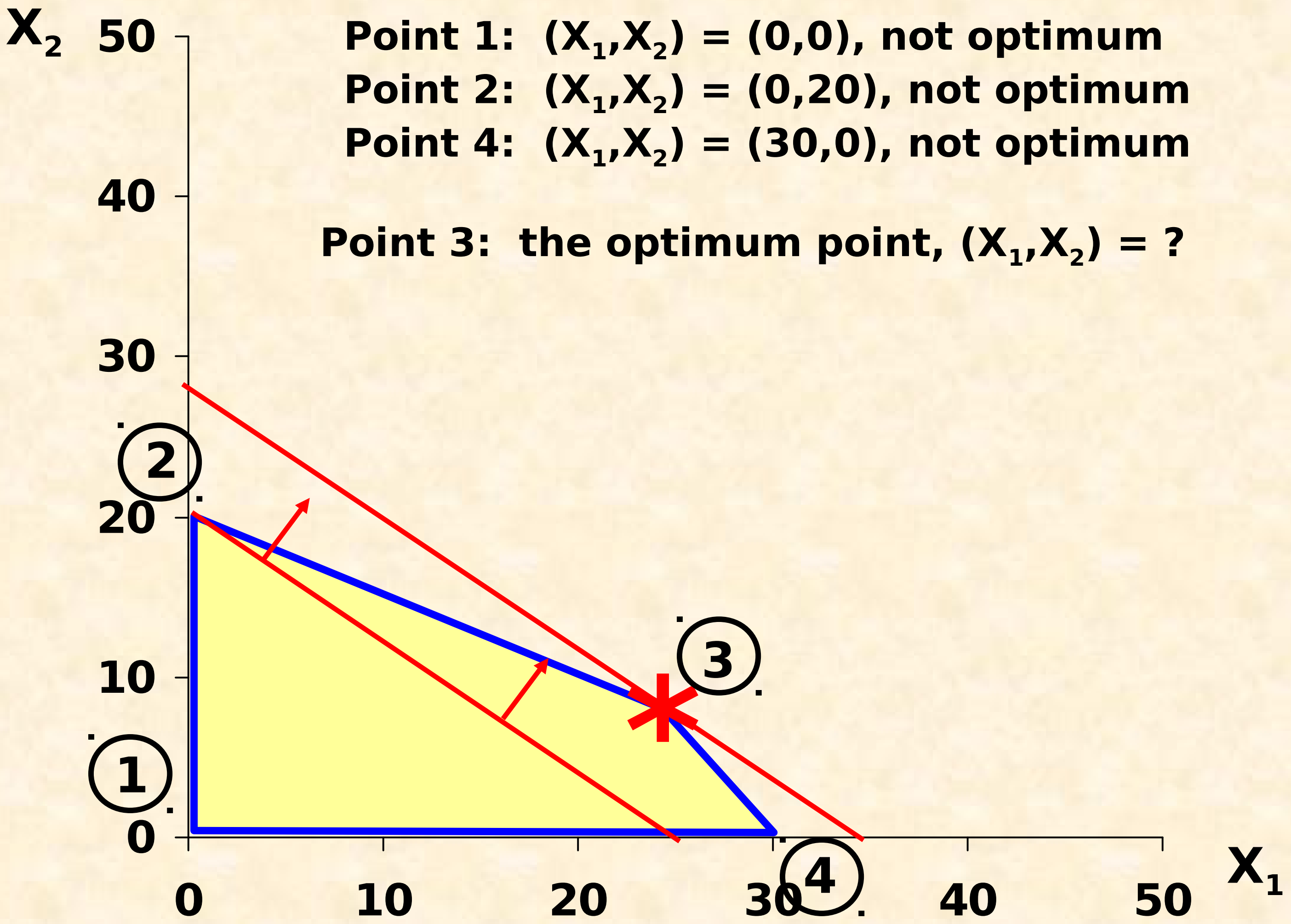
$$Z = 40X_1 + 50X_2$$

$$(40)(50) = 2000; \text{ pick a multiple of } 2000$$

$$\text{Set } Z = (0.5)(2000) = 1000$$

$$\text{If } X_1 = 0, X_2 = 20$$
$$\text{If } X_2 = 0, X_1 = 25$$

Draw Isoprofit Lines
Solve for Optimum X_1, X_2



Mathematical Solution

$X_1 + 2X_2 \leq 40$

$X_1 + 2X_2 = 40$

Solve for X_1

$X_1 = 40 - 2X_2$

$4X_1 + 3X_2 \leq 120$

$4X_1 + 3X_2 = 120$

Solve for X_1

$X_1 = (120 - 3X_2)/4$

At intersection, values of X_1 are equal:

$$40 - 2X_2 = (120 - 3X_2)/4$$

$$160 - 8X_2 = 120 - 3X_2$$

$$40 = 5X_2$$

$$X_2 = 8$$

$$X_1 = 40 - 2X_2$$

$$X_1 = 40 - 2(8)$$

$$X_1 = 24$$

$$Z = \$40(24) + \$50(8) = \$1360$$

The Holiday Meal Turkey Ranch

The Holiday Meal Turkey Ranch is considering purchasing 2 different brands of turkey feed which it will blend to provide a good, low-cost diet for its turkeys. Each feed contains, in varying proportions, some or all of 3 essential nutrients. Each pound of brand 1 contains 5 ounces of ingredient A, 4 ounces of ingredient B, and 1/2 ounce of ingredient C. Each pound of brand 2 contains 10 ounces of ingredient A, 3 ounces of ingredient B, and 0 ounces of ingredient C. Brand 1 feed costs 2 cents/lb while brand 2 feed costs 3 cents/lb.

Using LP, determine the lowest-cost diet that meets the following minimum monthly intake requirements for each nutritional ingredient.

Ingredient	Minimum Monthly Requirement (Per Turkey)
A	90
B	48
C	1.5

Decision Variables

What does management control?

Decision Variables

What does management control?

X_1 = pounds of brand 1 feed purchased

X_2 = pounds of brand 2 feed purchased

Objective Function

What is meant by best?

Objective Function

What is meant by best?

Best = minimum cost

- Cost per pound of brand 1 = 2 cents, Cost per pound of brand 2 = 3 cents
- # of pounds of brand 1 to purchase = X_1, # of pounds of brand 2 to purchase = X_2

- **Cost Function:**

$$Z = \$.02X_1 + \$.03X_2$$

- **Objective Function:**

Minimize $Z = \$.02X_1 + \$.03X_2$

Resource Constraints

What constraints must be obeyed?

What resources are limited?

Resource Constraints

Ingredient A

- **Minimum required: ?**

- **Amount supplied by each brand: ?**

- **Ingredient A constraint: ?**

Resource Constraints

Ingredient A

- Minimum required: 90 ounces per turkey per month

- Amount supplied by each brand: 5 ounces per pound of Brand 1, 10 ounces per pound of Brand 2

- Ingredient A constraint: $5X_1 + 10X_2 \geq 90$

Resource Constraints

Ingredient B

- **Minimum required: ?**

- **Amount supplied by each brand: ?**

- **Ingredient A constraint: ?**

Resource Constraints

Ingredient B

- **Minimum required: 48 ounces per turkey per month**

- **Amount supplied by each brand: 4 ounces per pound of Brand 1, 3 ounces per pound of Brand 2**

- **Ingredient B constraint:** $4X_1 + 3X_2 \geq 48$

Resource Constraints

<u>Ingredient C</u>

- <u>Minimum</u> required: ?
- Amount supplied by each brand: ?
- Ingredient C constraint: ?

Resource Constraints

Ingredient C

- **Minimum required: 1.5 ounces per turkey per month**

- **Amount supplied by each brand: 0.5 ounces per pound of Brand 1, 0 ounces per pound of Brand 2**

- **Ingredient B constraint: $0.5X_1 \geq 1.5$**

Resource Constraints

Non-negativity

- No negative quantities can be produced
- $X_1, X_2 \geq 0$

Model Summary

X_1= pounds of brand 1 feed purchased

X_2= pounds of brand 2 feed purchased

$$\text{Minimize } Z = 2X_1 + 3X_2$$

subject to

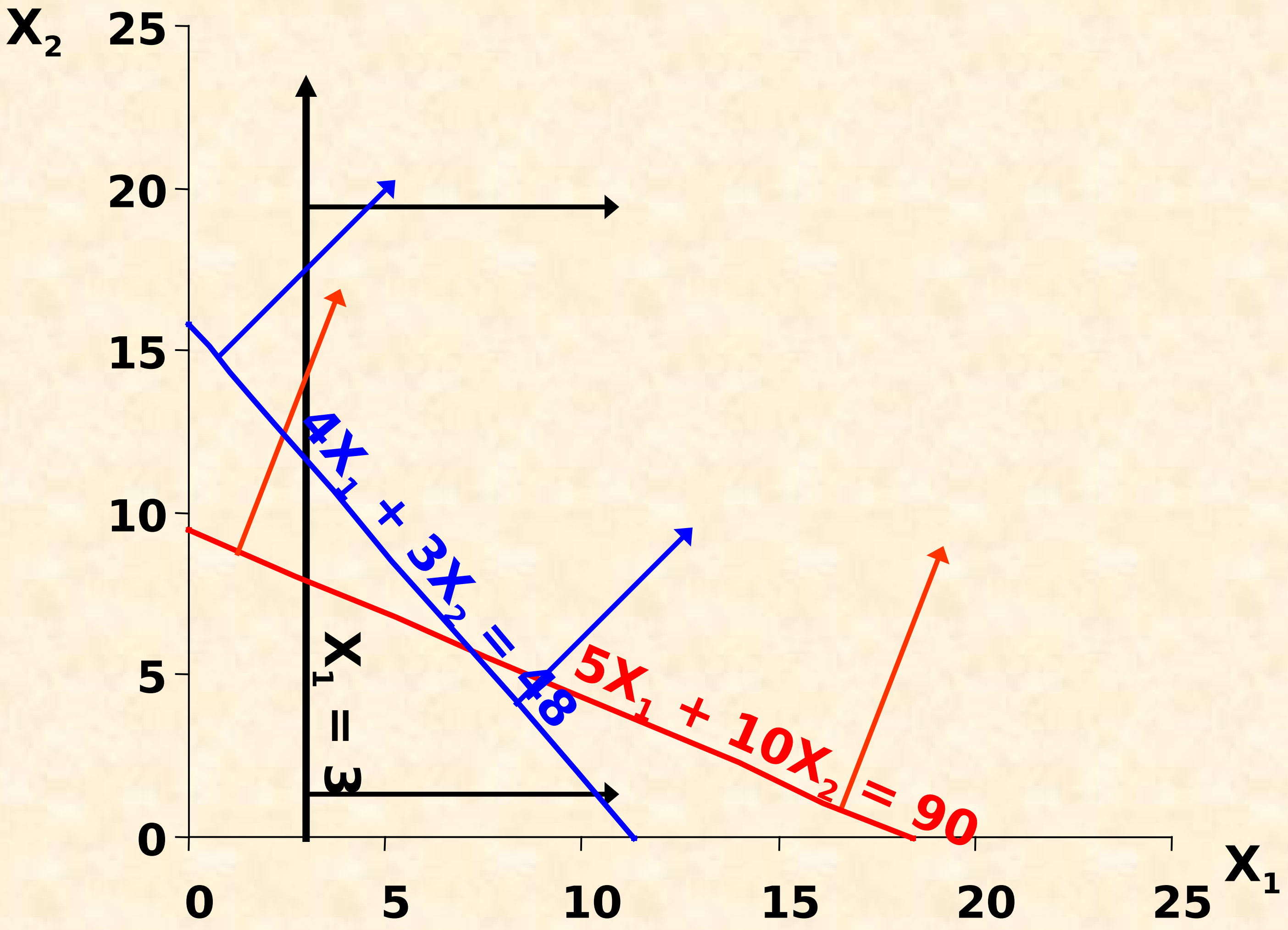
$$5X_1 + 10X_2 \geq 90$$

$$4X_1 + 3X_2 \geq 48$$

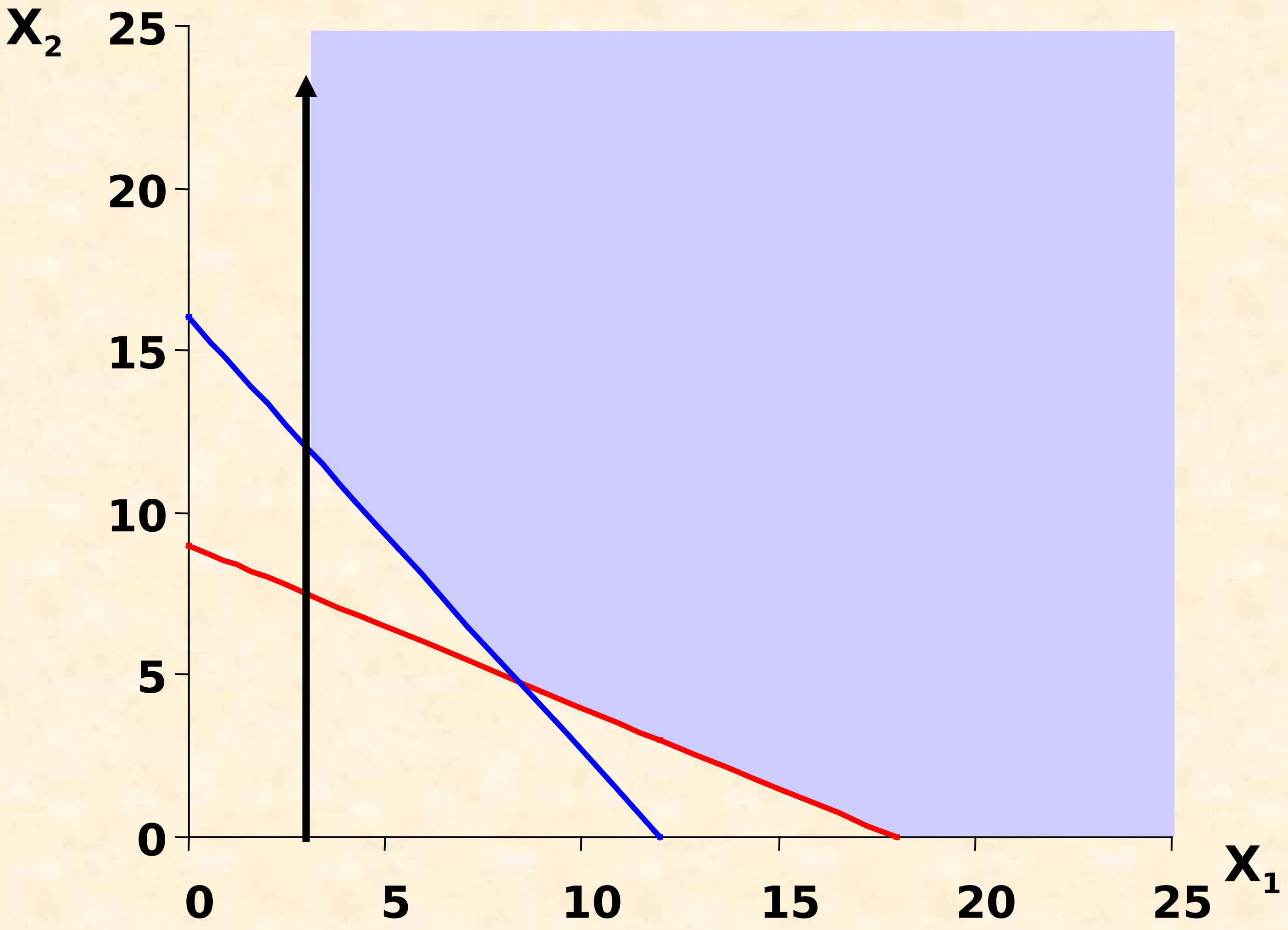
$$0.5X_1 \geq 1.5$$

$$X_1, X_2 \geq 0$$

Feasible Solutions



Feasible Region



Draw Iso*cost* Lines Solve for Optimum X_1, X_2

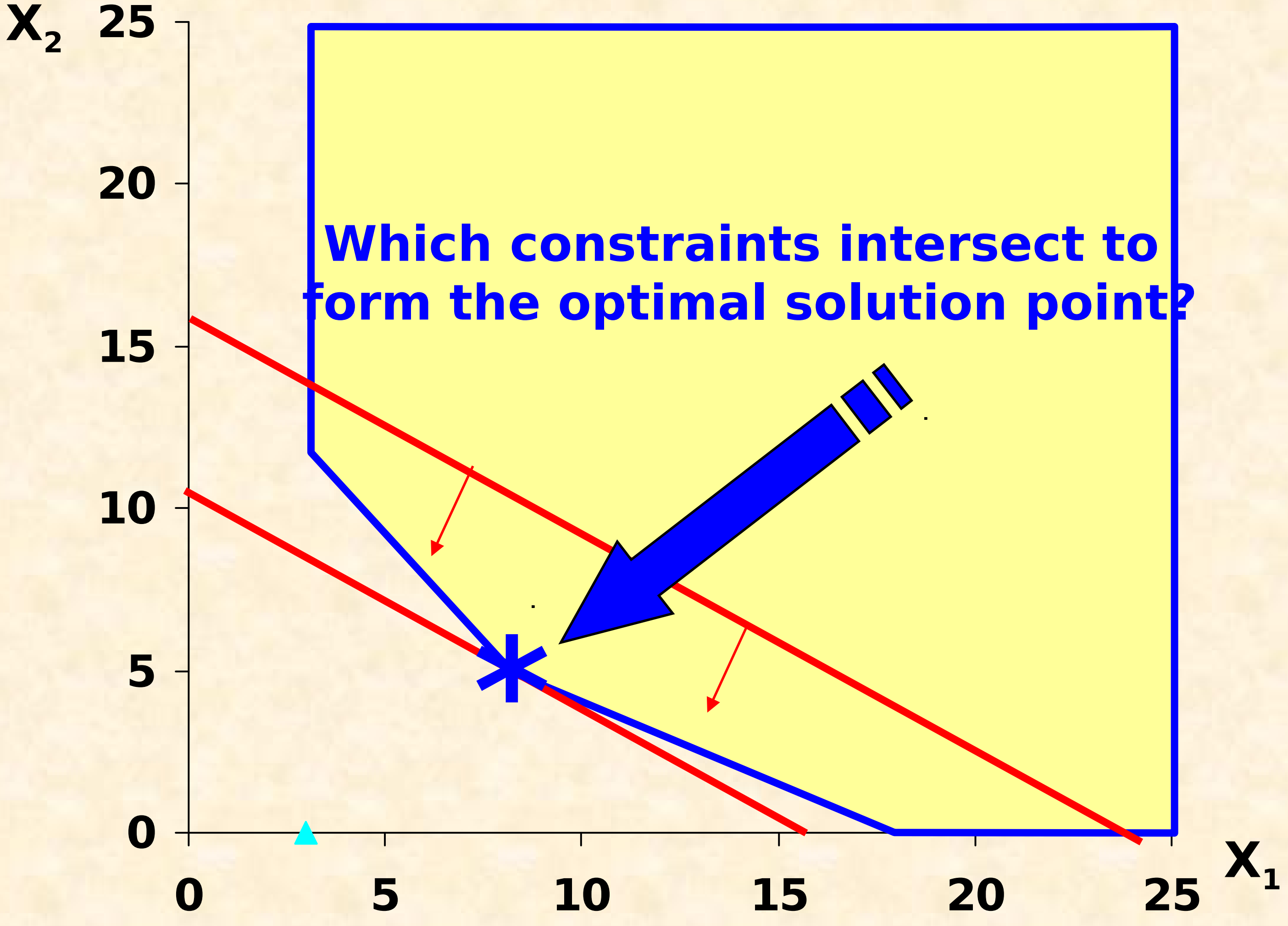
$$Z = 2X_1 + 3X_2$$

$$(2)(3) = 6; \text{pick a multiple of } 6$$

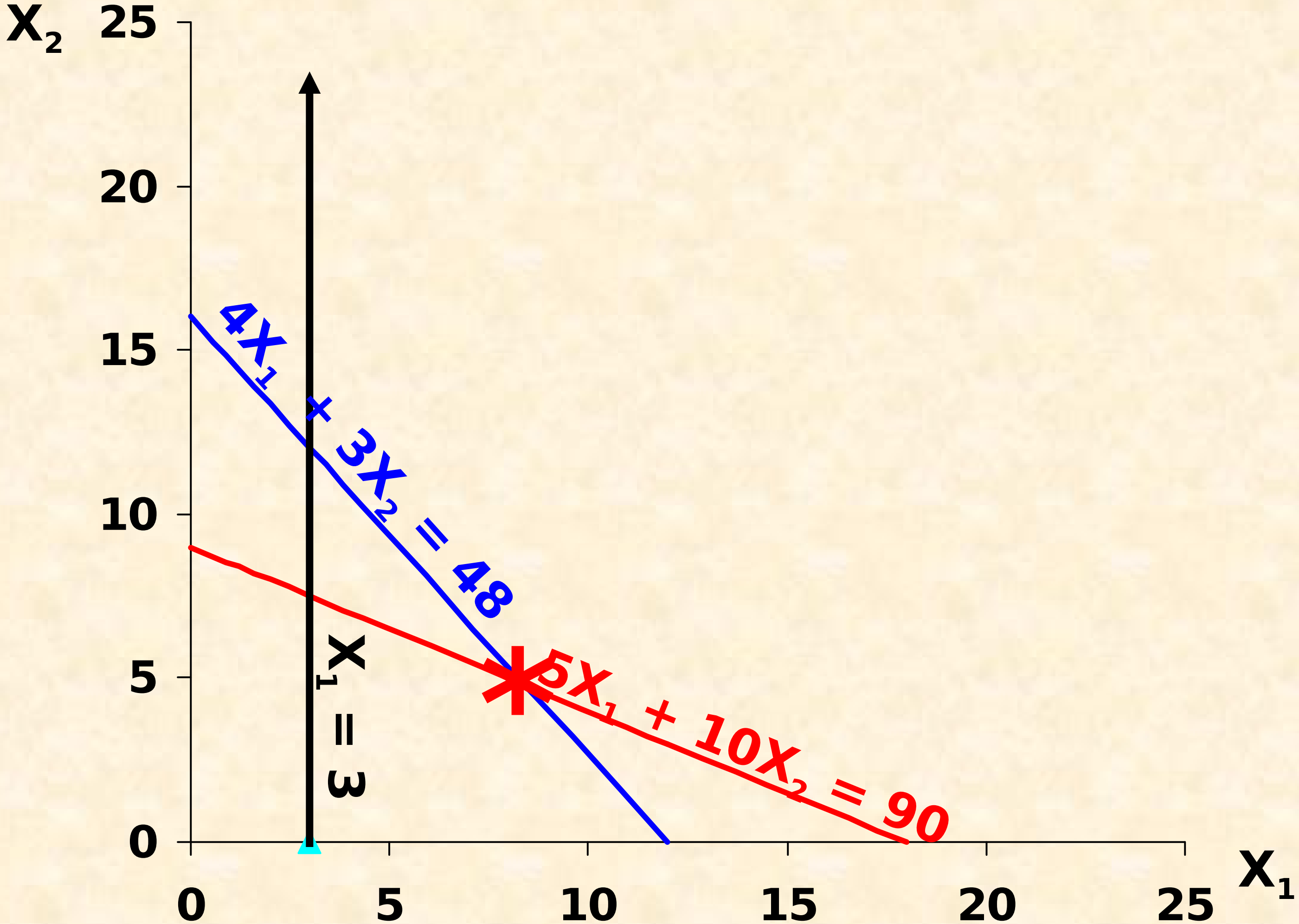
$$\text{Set } Z = (8)(6) = 48$$

$$\text{If } X_1 = 0, X_2 = 16$$
$$\text{If } X_2 = 0, X_1 = 24$$

Draw Isocost Lines
Solve for X_1, X_2



Feasible Solutions



Mathematical Solution

$5X_1 + 10X_2 \geq 90$

$5X_1 + 10X_2 = 90$

Solve for X_1

$X_1 = 18 - 2X_2$

$4X_1 + 3X_2 \geq 48$

$4X_1 + 3X_2 = 48$

Solve for X_1

$X_1 = (48 - 3X_2)/4$

At intersection, values of X_1 are equal:

$$18 - 2X_2 = (48 - 3X_2)/4$$

$$72 - 8X_2 = 48 - 3X_2$$

$$24 = 5X_2$$

$$X_2 = 4.8$$

$$X_1 = 18 - 2X_2$$

$$X_1 = 18 - 2(4.8)$$

$$X_1 = 8.4$$

$$Z = 2(8.4) + 3(4.8) = 16.8 + 14.4 =$$

Irregular Problems

- **Infeasibility**
  - **A condition that arises when there is no solution to a linear programming problem that satisfies all constraints simultaneously**
- **Unbounded Problems**
  - **A linear program that does not have a finite solution**
- **Redundancy**
  - **Constraints are present that do not affect the feasible region**
- **Multiple Optimal Solutions**
  - **Occurs when a problem's isoprofit or isocost line exactly parallels a resource constraint line at the optimum solution point**

An Infeasible Problem

$$\text{Maximize } Z = 5X_1 + 3X_2$$

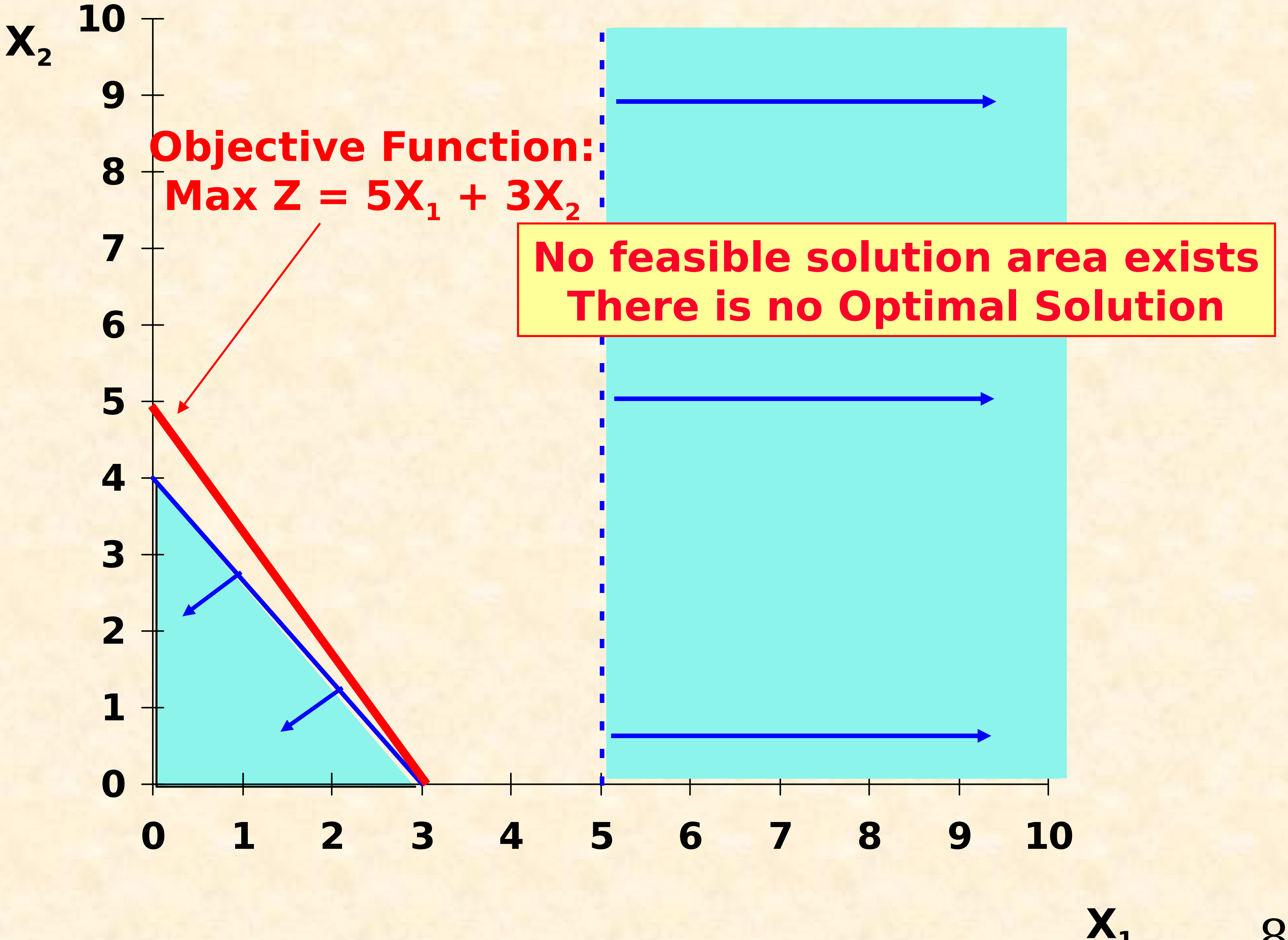
subject to

$$4X_1 + 3X_2 \leq 12$$

$$X_1 \geq 5$$

$$X_1, X_2 \geq 0$$

An Infeasible Problem



Infeasible Problems

- Rarely occur
- Usually a result of improper formulation

An Unbounded Problem

$$\text{Maximize } Z = 4X_1 + 4X_2$$

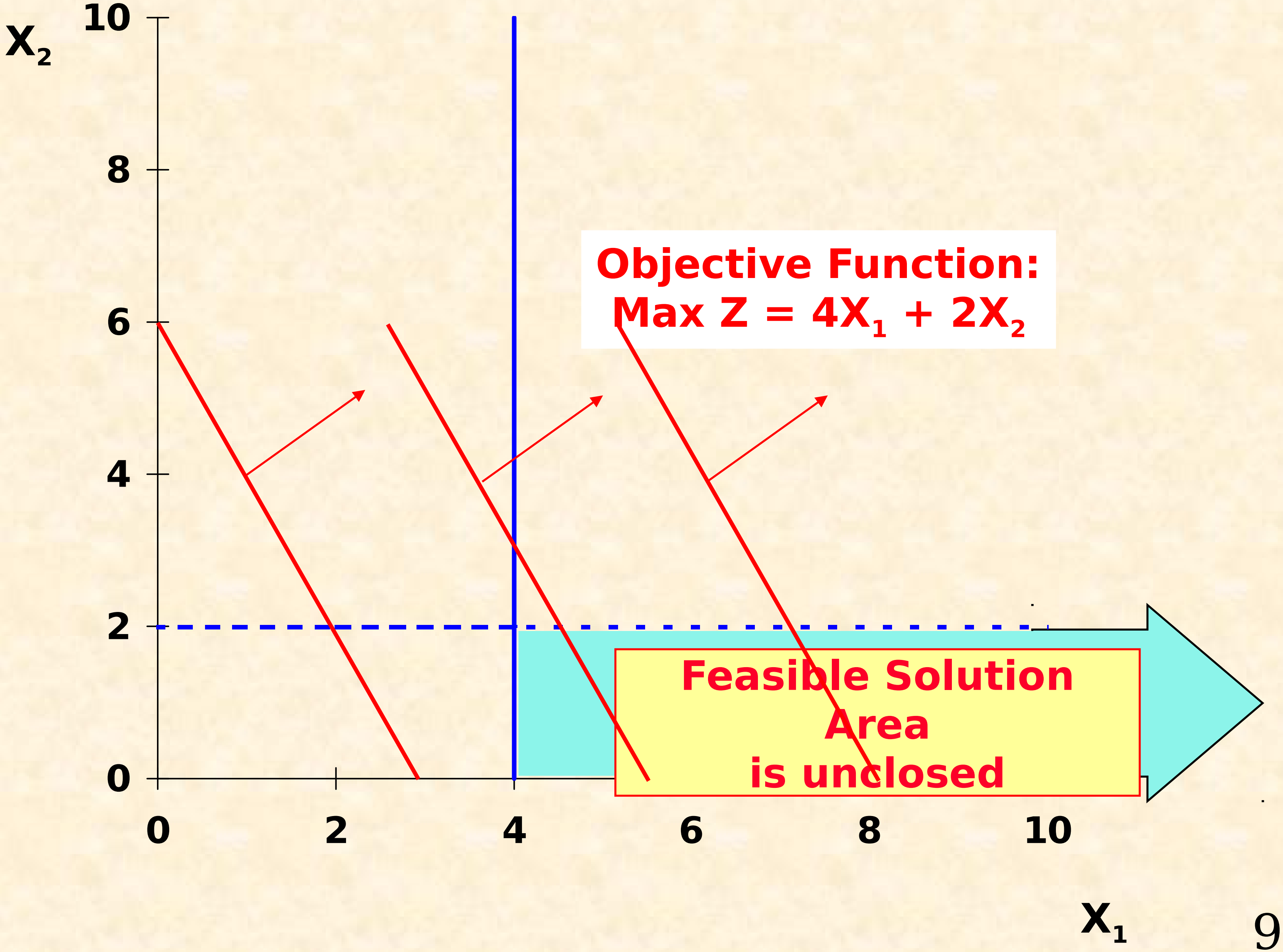
subject to

$$X_1 \geq 4$$

$$X_2 \leq 2$$

$$X_1, X_2 \geq 0$$

An Unbounded Problem



Unbounded Problems

- The feasible solution area is not closed
- Objective function can be increased indefinitely without reaching a maximum value
- An “impossible” situation in a world of limited resources
  - A constraint has likely been omitted

Redundancy

Maximize $Z = 1X_1 + 2X_2$

subject to

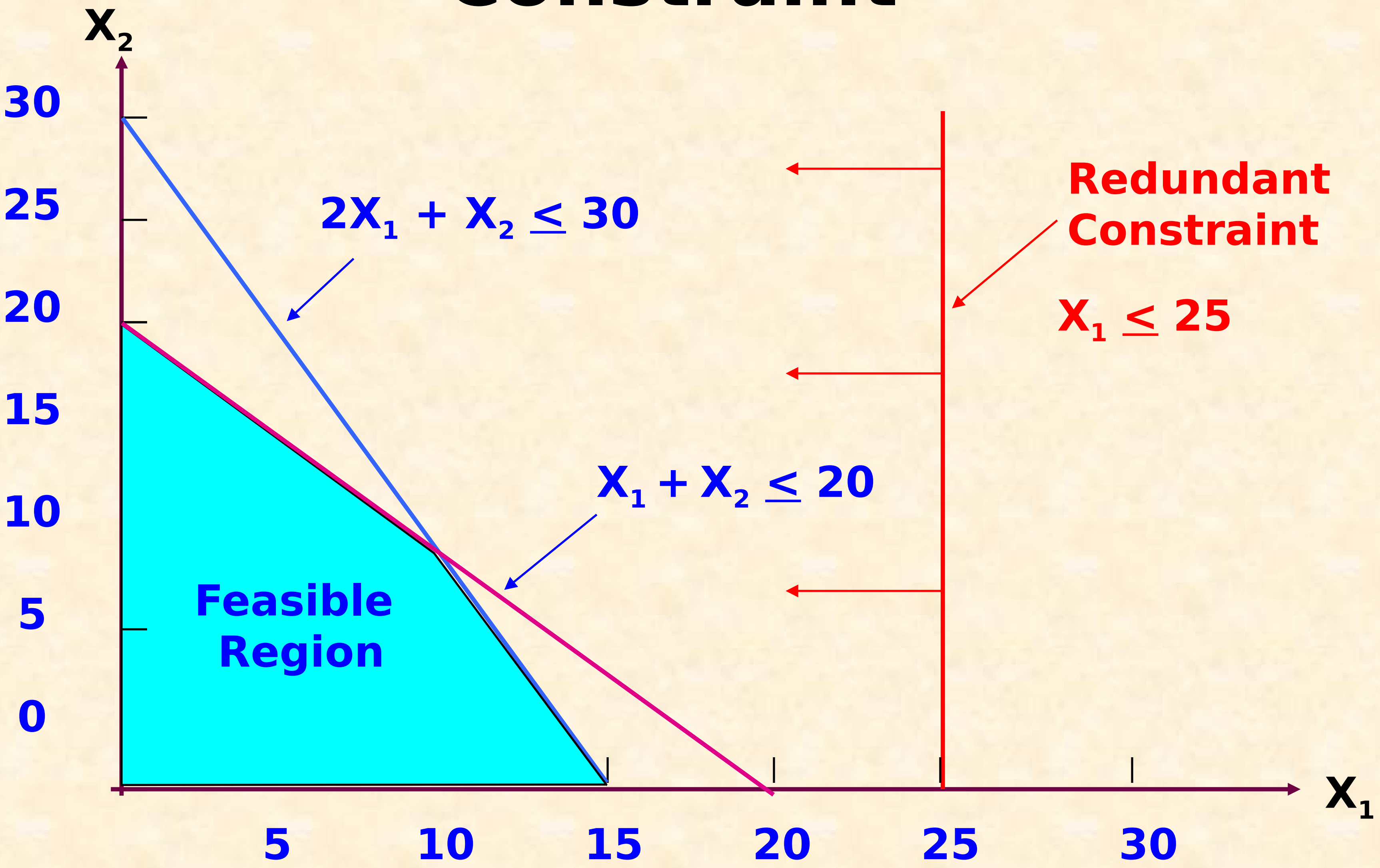
$$X_1 + X_2 \leq 20$$

$$2X_1 + X_2 \leq 30$$

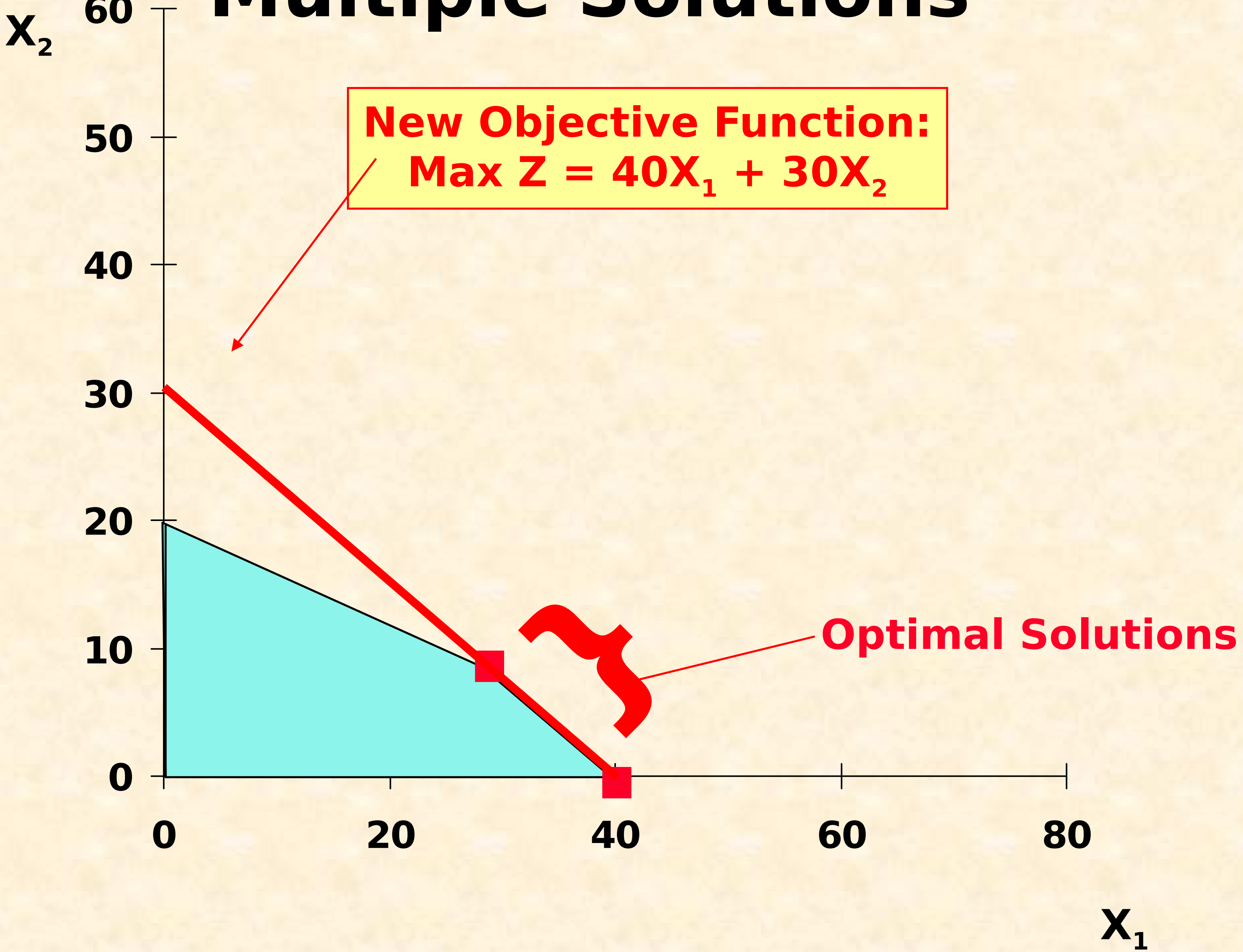
$$X_1 \leq 25$$

$$X_1, X_2 \geq 0$$

A Problem with a Redundant Constraint



Multiple Solutions



Multiple Solutions

- **Objective function parallels a constraint line at the optimal solution**

- **Any point along the line segment is optimal**

- **Endpoints are called alternate optimal solutions**

Sensitivity Analysis

- **Solution process thus far has been deterministic**
  - **Have assumed models parameters known with certainty**
    - » **prices are fixed, resources are known, etc.**
- **Unlikely due to real world dynamics**
  - **It is extremely important to know how sensitive the optimal solution is to changes in the model's assumptions or data**
- **With the advent of desktop computers and linear programming software packages, sensitivity analysis often is done by simply changing a model's parameters and rerunning the model to assess the impact of the changes**

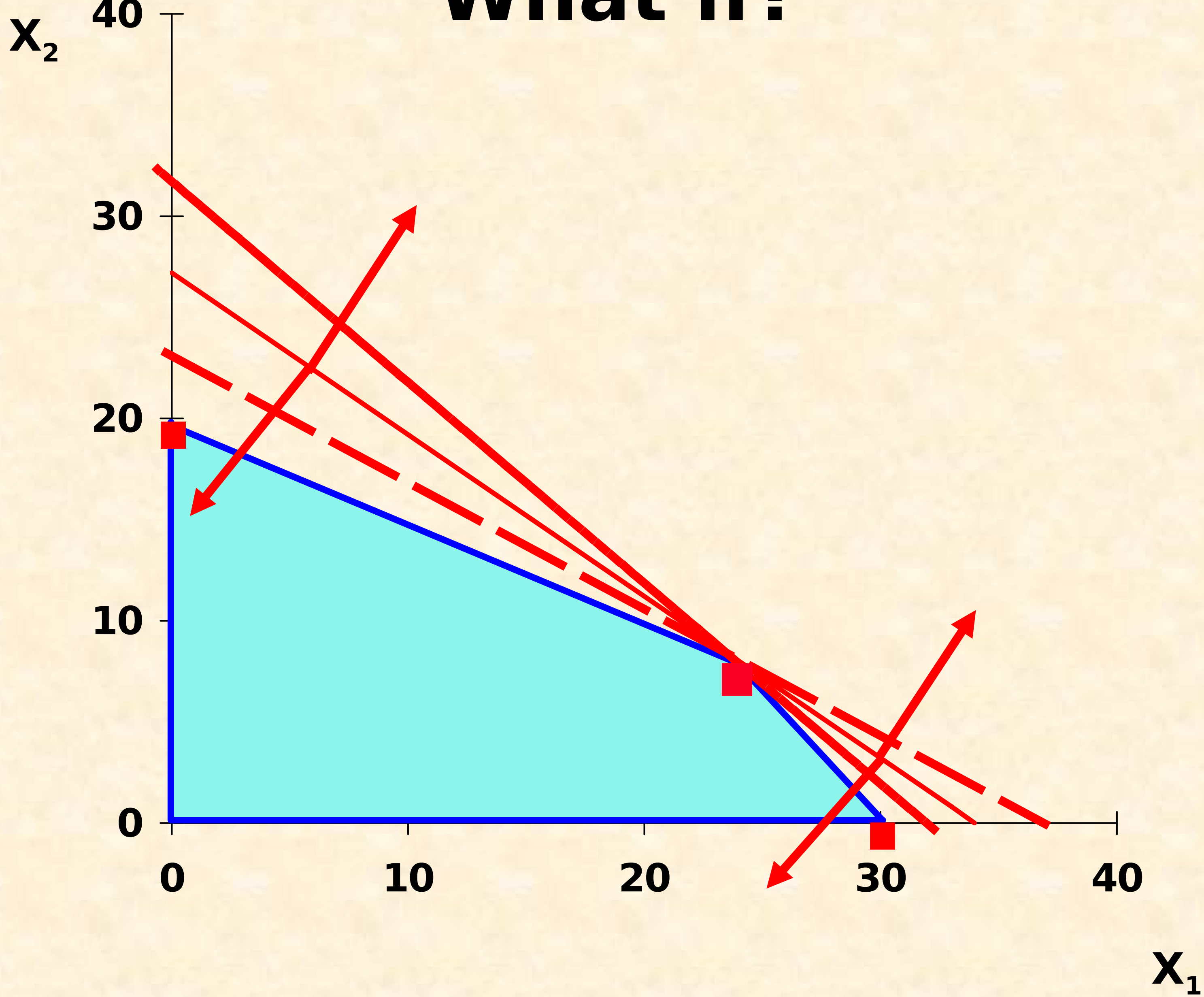
Sensitivity Analysis

- **Sensitivity analysis is generally performed to examine a model's sensitivity to changes in three important areas**
  - **Objective function coefficients**
    - » **measure contribution rates for decision variables**
  - **Constraint function coefficients**
    - » **measures of technology**
  - **Constraint function right hand side values**
    - » **measures of resource availability**

Objective Function Coefficients

- **A product's contribution to profit or cost fluctuates periodically, as do most of a firm's expenses**
- **Graphical implication is that, while the feasible solution area remains exactly the same, the slope of the isoprofit or isocost line changes**
  - **Optimal solution point may occur at a different extreme point**
    - » **a different combination of decision variables may occur**
    - » **maximum profit/minimum cost may change**

What if?



Resource Constraint Coefficients

- **Resource constraint coefficients often reflect changes in the state of technology**
  - **Improvements in technology may improve efficiency of the production process so that fewer resources are required to generate the same level of output**
- **Graphical implication is that, while the slope of the isoprofit or isocost line does not change, the feasible solution area may change significantly**
  - **A new or different corner point may become optimal**
  - **Maximum profit/minimum cost may change**

Right Hand Side Values

- **Right hand side values of constraints are indicative of the resources available to a firm**

- **Graphical implication is a change in the feasible solution area**
  - **Maximum profit/minimum cost may change**

Computer Solutions

- Graphing limited to two decision variables
  - Not a practical limitation

- Computers offered a powerful tool for number crunching
  - Proper formulation a prerequisite

Proper Formulation = Standard Form

- **Tremendous diversity of LP problems posed a problem for software developers**
  - **Minimize, maximize**
  - **All $\geq$ constraints, all $\leq$ constraints, combinations**

- **Standard Form was developed to reduce the possible variations of LP problems to one common model**

Standard Form

- **Convert inequalities to equations**
  - **Ready to solve simultaneous equations for optimal solution**
  - **Converting to constraints that look the same simplified formulation and solution by computer**

Transformation to Standard Form

$\leq$ Constraints

- **Add a Slack Variable (S_i)**
  - **Free to assume any value necessary to make the left-hand side (LHS) of the equation equal to the right-hand side(RHS)**
  - **Represent unused resources, or *slack* in a resource constraint**

$X_1 + 2X_2 \leq 40$ hrs of labor → **$X_1 + 2X_2 + S_1 = 40$**

$4X_1 + 3X_2 \leq 120$ lbs of clay → **$4X_1 + 3X_2 + S_2 = 120$**

For $(X_1,X_2) = (5,10)$

$S_1 = 40 - 5 - 2(10)$

$S_1 = 15$ unused hrs of labor

$S_2 = 120 - 4(5) -3(10)$

$S_2 = 70$ unused lbs of clay

Slack Variables

- Unused resources contribute nothing to profit
  - No effect on the objective function

- Nonnegative
  - Negative resources not possible

Transformation to Standard Form

$\geq$ Constraints

- **Subtract** a Surplus Variable (S_i)
  - Free to assume any value necessary to make the left-hand side (LHS) of the equation equal to the right-hand side(RHS)
  - Reflect the excess above a minimum requirement level, or ***surplus*** in a resource constraint

$5X_1 + 10X_2 \geq 90$ oz of Ingred A

$4X_1 + 3X_2 \geq 48$ oz of Ingred B

$5X_1 + 10X_2 - S_1 = 90$

$4X_1 + 3X_2 - S_2 = 48$

For $(X_1,X_2) = (0,20)$

$S_1 = 5(0) + 10(20) - 90$

$S_1 = 110$ excess oz of Ingred A

$S_2 = 4(0) + 3(20) - 48$

$S_2 = 12$ excess oz of Ingred B

Surplus Variables

- **Excess resources contribute nothing to cost**
  - **A “fringe benefit” with no effect on objective function**

- **Nonnegative**
  - **Negative resources not possible**

Computer Solution

- **Based on the Simplex Method**
  - **Requires standard form**
- **Computer software incorporates the mathematical steps of the simplex method**
  - **Software evolution = constraints automatically converted to equalities**
- **Slack and surplus variables are important elements of *solver output***
  - **Offer management significant insight**
- **Solution Analysis**

Proper Formulation Essential

- **Read the story**
- **Identify the Problem**
  - **Decision variables**
  - **Objective function**
  - **Model constraints**
- **Formulate the model**
- **Convert to spreadsheet format compatible with computer solver**
- **Solve problem then analyze solution**

Spreadsheet Models

Principle Elements

- **Numbers**
  - **Decision Variables**
  - **Parameters**

- **Formulas**
  - **Functional relationships between variables and parameters**

Numbers

- Decision Variables
  - Numbers under managerial control
  - Adjustable/changeable cells

- Parameters
  - Not under managerial control
    - » objective function coefficients
      - profit per unit
    - » right hand side values
      - available inventory of resources
    - » resource constraint coefficients
      - resource requirements per unit

Spreadsheet Formulas

- **<u>KEEP PARAMETER DATA AND FORMULAS SEPARATE</u>**
  - **Eliminates need to modify formulas when parameters change**

- **SUMPRODUCT**
  - **Excel function that yields the inner product of two vectors**

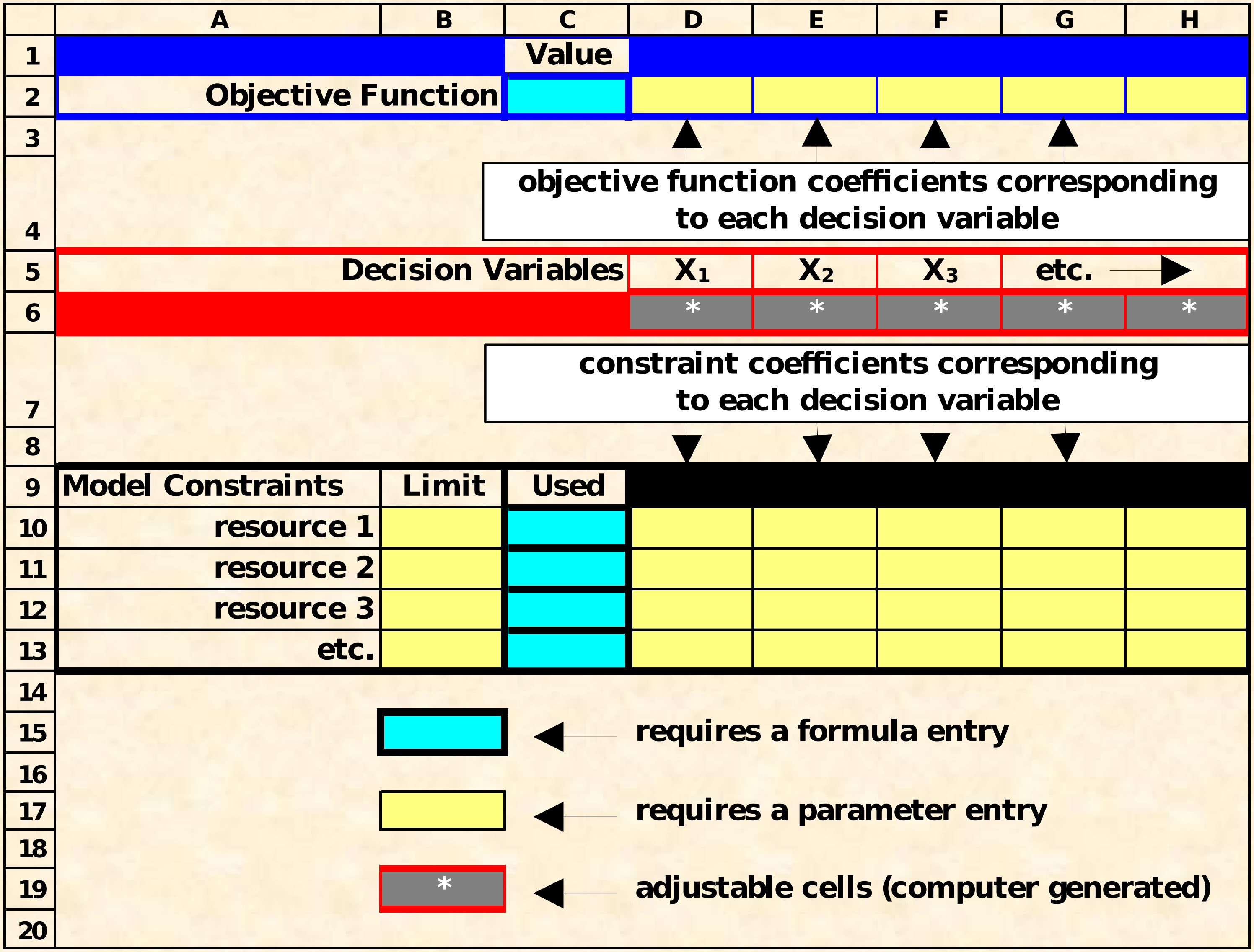
SUMPRODUCT

- **Profit Function: $Z = 5X_1 + 6X_2 + 8X_3$**
- **Decision Variable Quantities: $X_1 = 2, X_2 = 6, X_3 = 4$**

$$\text{Total Profit} = \sum_{i=1}^{3} (\text{Profit}_i)(\text{Quantity}_i)$$
$$= (5)(2) + (6)(6) + (8)(4) = 78$$

	A	B	C	D
1	Vector Notation			
2		X_1	X_2	X_3
3	Objective Function Coefficients	5	6	8
4	Decision Variable Quantities	2	6	4
5				
6	=SUMPRODUCT(B3:D3,B4:D4)			

LP Model Formulation for Microsoft Excel



Proper Constraint Form

- **Variables on LHS, Numeric Coefficients**
- **Numerical Values on RHS**
  - **No "negative resources"**
- **Equalities**

Proper form for the following:

- **Production of product 3 must be as much or more than the production of products 1 and 2**

$$X_3 \geq X_1 + X_2$$

- **Production of product 1 must be no more than twice the production of products 2 and 3**

$$\frac{X_1}{X_2 + X_3} \leq 2$$

Proper Constraint Form

$$X_3 \geq X_1 + X_2$$

- **Variables on LHS, Numerical Values on RHS**

$$X_3 - X_1 - X_2 \geq 0$$

- **Equality**

$$X_3 - X_1 - X_2 - S_1 = 0$$

What if constraint was of form $X_3 - X_1 - X_2 \geq -4$?

$$X_3 - X_1 - X_2 - S_1 = -4$$

RHS must be $\geq$ 0; convert by multiplying both sides by -1

$$-1(X_3 - X_1 - X_2 - S_1) = -1(-4)$$

$$X_1 + X_2 + S_1 - X_3 = 4$$

Proper Constraint Form

$$\frac{X_1}{X_2 + X_3} \leq 2$$

- Variables on LHS, Numerical Values on RHS

$$\frac{X_1}{X_2 + X_3} \leq 2$$

  - Relationship of variables/lack of numeric coefficients still a problem

$$X_1 \leq 2(X_2 + X_3)$$

$$X_1 - 2X_2 - 2X_3 \leq 0$$

- Equality

$$X_1 - 2X_2 - 2X_3 + S_2 = 0$$